

38th United States of America Mathematical Olympiad

1. **Solution 1.** Let ω denote the circumcircle of P, Q, R, S and let O denote the center of ω . Line XY is the radical axis of circles ω_1 and ω_2 . It suffices to show that O has equal power to the two circles; that is, to show that

$$OO_1^2 - O_1S^2 = OO_2^2 - O_2Q^2 \quad \text{or} \quad OO_1^2 + O_2Q^2 = OO_2^2 + O_1S^2.$$

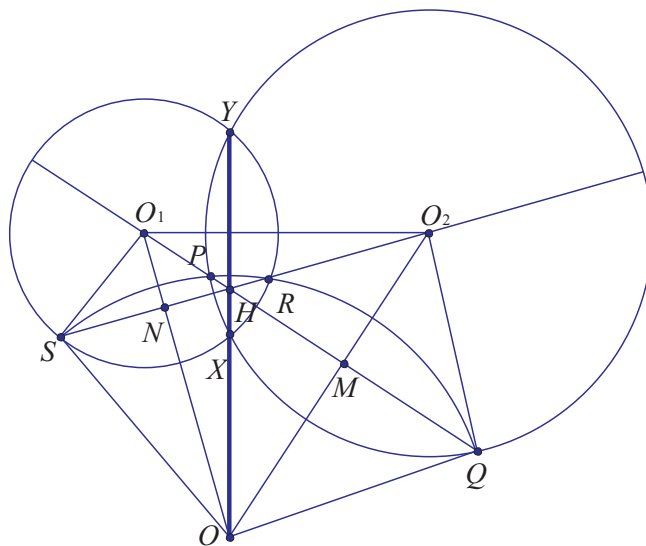
Let M and N be the intersections of lines O_2O, ℓ_1 and O_1O, ℓ_2 . Because circles ω and ω_2 intersect at points P and Q , we have $PQ \perp OO_2$ (or $\ell_1 \perp OO_2$). Hence

$$OO_1^2 - OQ^2 = (OM^2 + MO_1^2) - (OM^2 + MQ^2) = (O_2M^2 + MO_1^2) - (O_2M^2 + MQ^2) = O_2O_1^2 - O_2Q^2$$

or

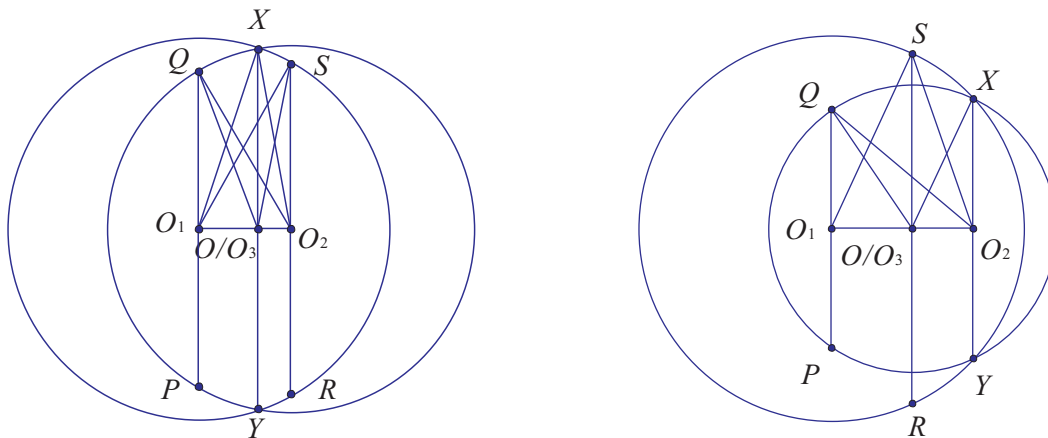
$$O_2O_1^2 + OQ^2 = OO_1^2 + O_2Q^2.$$

Likewise, we have $O_2O_1^2 + OS^2 = OO_2^2 + O_1S^2$. Because $OS = OQ$, we obtain that $OO_1^2 + O_2Q^2 = OO_2^2 + O_1S^2$, which is what to be proved.



Solution 2. We maintain the notations of the first solution. Three pairs of circles (ω, ω_1) , (ω_1, ω_2) , (ω_2, ω) meet at three pairs of points (R, S) , (X, Y) , (P, Q) , respectively; that is, lines RS, XY, PQ are the respective radical axes of these pairs of circles. We consider two cases.

In the first case, we assume that these three radical axes are not parallel. They must be concurrent at the radical center, denoted by H , of these three circles. In particular, it follows that H, X, Y lie on a line, denoted by ℓ , and $\ell \perp O_1O_2$. On the other hand, $O_1M \perp O_2O$ and $O_2N \perp O_1O$. Hence H is the orthocenter of triangle OO_1O_2 , from which it follows that $OH \perp O_1O_2$. Therefore, O lies on ℓ ; that is, X, P, Q are collinear.



In the second case, we assume that these three radical axes are parallel. We will then deduce the above configurations. Let O_3 be the midpoint of segment XY . From right triangles $O_1O_3Q, O_1O_3X, O_1O_2Q$, we have

$$O_3Q^2 = O_1Q^2 + O_1O_3^2 = O_2Q^2 - O_1O_2^2 + O_1X^2 - XO_3^2,$$

which is an expression symmetric about circles ω_1 and ω_2 . Hence we can easily obtain that $O_3Q^2 = O_3S^2$ and that O_3 is the circumcenter of isosceles trapezoid $PQSR$; that is, $O_3 = O$, completing the proof.

This problem was suggested by Ian Le. The solutions were contributed by Zuming Feng.

2. The maximum size is n if n is even, and $n + 1$ if n is odd, achieved by the subset

$$\{-n, \dots, -\lfloor \frac{n}{2} \rfloor - 1, \lfloor \frac{n}{2} \rfloor + 1, \dots, n\}.$$

Lemma. Let A, B be finite nonempty subsets of \mathbb{Z} . Then the set $A + B = \{a + b : a \in A, b \in B\}$ has cardinality at least $|A| + |B| - 1$.

Proof. Write $A = \{a_1, \dots, a_l\}$ and $B = \{b_1, \dots, b_m\}$ with $a_1 < \dots < a_l$ and $b_1 < \dots < b_m$. Then

$$a_1 + b_1, \dots, a_1 + b_m, a_2 + b_m, \dots, a_l + b_m$$

is a strictly increasing sequence of $l + m - 1$ elements of $A + B$. □

Let S be a subset of $\{-n, \dots, n\}$ with the desired property; clearly $0 \notin S$. Put $A = S \cap \{-n, \dots, -1\}$ and $B = S \cap \{1, \dots, n\}$. Then $A + B$ and $-S = \{-s : s \in S\}$ are disjoint subsets of $\{-n, \dots, n\}$, so by the lemma,

$$2n + 1 \geq |A + B| + |-S| \geq |A| + |B| - 1 + |S| = 2|S| - 1,$$

or $|S| \leq n + 1$. If n is odd, we are done.

If n is even, we must still show that $|S| = n + 1$ is impossible. Since $A + B \subseteq \{-n + 1, \dots, n - 1\}$, we cannot achieve the equality $2n + 1 = |A + B| + |-S|$ unless $-n, n \in -S$, or equivalently $-n, n \in S$. Since $-n \in S$, each of the sets $\{1, n - 1\}, \dots, \{n/2 - 1, n/2 + 1\}, \{n/2\}$ must contain an element not in B . Thus $|B| \leq n/2$, and similarly $|A| \leq n/2$, contradicting the hypothesis $|S| = n + 1$.

This problem was suggested by Kiran Kedlaya with Tewodros Amdeberhan.

3. **a)** We prove the first part by induction on the number n of dominoes in the tiling. The claim is clearly true for $n = 1$. So suppose we have a chessboard polygon that can be tiled by $n > 1$ dominoes. Of all the leftmost squares in the polygon, select the lowest one and label it L ; assume for sake of argument that square L is black. In the given tiling, remove the domino covering L , leaving a polygon which may be tiled with $n - 1$ dominoes. By the induction hypothesis, this chessboard polygon can be tastefully tiled.

Now replace the domino that was removed. If this domino is horizontal, then we are guaranteed that the augmented tiling is still tasteful, since square L is black and there are no squares below it. If the domino is vertical the augmented tiling may still be tasteful, but if not the trouble can only arise because there is another vertical domino directly to its right. In this case rotate the offending pair of dominoes to get two horizontal dominoes. We are not done yet, but if we now repeat this process—removing the *horizontal* domino covering L , tiling the remainder, and replacing the domino—then we will obtain a tasteful tiling.

If square L is white we may obtain a tasteful tiling by performing a similar process. This time we only encounter difficulty if the domino covering L in the original tiling is horizontal, in which case there must be another horizontal domino directly above it. We rotate this pair, remove the now vertical domino covering L , tile the remainder tastefully using the induction hypothesis, and restore the vertical domino to finish.

b) Suppose now that there are two tasteful tilings of a given chessboard polygon. By overlaying these two tilings we obtain chains of overlapping dominos, since every square is part of one domino from each tiling. For example, a chain of length one indicates a domino common to both tilings. A chain of length two cannot occur, since these arise when a 2×2 block is covered by horizontal dominos in one tiling and vertical dominos in the other, and one of these configurations will be distasteful.

Since the tilings are distinct a chain of length three or more must occur; let R be the region consisting of such a chain along with its interior, if any. (It is possible that such a chain may completely occupy a region, so that only some of the dominoes in the chain adjoin squares outside of R .) Note that the chain must include a horizontal domino along its lowermost row. If there are two or more overlapping horizontal dominos, then one of them will be a WB domino, i.e. have a white square on the left. Otherwise there are two adjacent vertical dominos that overlap with the single horizontal domino; since they are part of a tasteful tiling we again must have a WB domino. We will now focus on the tiling that includes this WB domino.

The two squares above the WB domino must be part of region R . Furthermore, a single horizontal domino cannot cover them both, nor can a pair of vertical dominos. (Both cases yield distasteful configurations.) Hence a horizontal domino must cover at least one of these squares, extending past the given WB domino either to the left or right. Hence we can deduce the existence of a horizontal WB domino on the next row up. We may repeat this argument until we reach a horizontal WB domino in region R for which the two squares immediately above it are not both in region R . Hence this domino must be part of the chain that defined R .

Now imagine walking along the chain, starting on the white square of the WB domino that exists along the lowest row of region R and taking the first step towards the black square of the same domino. Draw an arrow along each domino in the direction of travel all the way around the chain. Since the squares must alternate white and black, these arrows will always point from a white square to a black square. Furthermore, since the interior of the region was initially to our left when we began the loop, it will always be to our left whenever the chain follows the boundary of R .

But we now reach a contradiction. We earlier deduced the existence of a horizontal WB domino that was part of the chain and was adjacent to the boundary of R , having a square above it that was not part of R . Hence this domino must be traversed from right to left, since we leave the interior of R to our left as we traverse the loop. Hence it must contain an arrow pointing to the left, implying that it must be a BW domino instead. This contradiction completes the proof.

This problem was suggested by Sam Vandervelde.

4. **Remark:** Let $m = \min(a_1, a_2, \dots, a_n)$ and $M = \max(a_1, a_2, \dots, a_n)$. By symmetry, we may assume without loss of generality, $m = a_1 \leq a_2 \leq \dots \leq a_n = M$. We present three

solutions. The first solution is a direct application of the Cauchy-Schwarz Inequality. The second solution bypasses Cauchy-Schwarz by applying one of its proofs. The third solution applies the AM-GM and AM-HM inequalities. All of them share the same finish, the case for $n = 2$.

If $n = 2$, given condition reads

$$(m + M) \left(\frac{1}{m} + \frac{1}{M} \right) \leq \frac{25}{4}.$$

It follows that

$$4(m + M)^2 \leq 25Mm \quad \text{or} \quad (4M - m)(M - 4m) \leq 0. \quad (1)$$

Because $4M - m > 0$, it must be that $M - 4m \leq 0$ and thus $M \leq 4m$.

We may assume from now that $n \geq 3$.

Solution 1. The Cauchy-Schwarz Inequality gives

$$\begin{aligned} \left(n + \frac{1}{2} \right)^2 &\geq (a_1 + a_2 + \cdots + a_n) \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right) \\ &= (m + a_2 + \cdots + a_{n-1} + M) \left(\frac{1}{M} + \frac{1}{a_2} + \cdots + \frac{1}{a_{n-1}} + \frac{1}{m} \right) \\ &\geq \left(\sqrt{\frac{m}{M}} + \underbrace{1 + \cdots + 1}_{n-2} + \sqrt{\frac{M}{m}} \right)^2. \end{aligned}$$

Hence

$$n + \frac{1}{2} \geq \sqrt{\frac{m}{M}} + n - 2 + \sqrt{\frac{M}{m}} \quad \text{or} \quad \sqrt{\frac{m}{M}} + \sqrt{\frac{M}{m}} \leq \frac{5}{2}. \quad (2)$$

It follows that

$$2(m + M) \leq 5\sqrt{Mm},$$

which is (1), completing our proof.

Solution 2. Consider the quadratic polynomial (in x)

$$\begin{aligned} p(x) &= \frac{1}{2} \left[\left(\sqrt{a_1}x + \frac{1}{\sqrt{a_n}} \right)^2 + \left(\sqrt{a_n}x + \frac{1}{\sqrt{a_1}} \right)^2 \right. \\ &\quad \left. + \sum_{i=2}^{n-1} \left(\sqrt{a_i}x + \frac{1}{\sqrt{a_i}} \right)^2 + \left(5 - 2\sqrt{\frac{m}{M}} - 2\sqrt{\frac{M}{m}} \right) x \right] \\ &= \left(\frac{1}{2} \sum_{i=1}^n a_i \right) x^2 + \frac{2n+1}{2} \cdot x + \left(\frac{1}{2} \sum_{i=1}^n \frac{1}{a_i} \right) \end{aligned}$$

Its discriminant is equal to

$$\Delta = \left(n + \frac{1}{2}\right)^2 - \left(\sum_{i=1}^n a_i\right) \left(\sum_{i=1}^n \frac{1}{a_i}\right),$$

which, by the given condition is nonnegative. Thus $p(x)$ has a real root r , and

$$0 = 2p(r) \geq \left(5 - 2\sqrt{\frac{m}{M}} - 2\sqrt{\frac{M}{m}}\right)r.$$

Because all of the coefficients of p are positive, we must have $r < 0$, from which (2) follows.

Solution 3. We set $a = \frac{a_2 + \dots + a_{n-1}}{n-2}$. Then $m \leq a_2 \leq a \leq a_{n-1} \leq M$ and $a_2 + \dots + a_{n-1} = (n-2)a$. By the AM-HM Inequality, we have

$$\frac{1}{a_2} + \dots + \frac{1}{a_{n-1}} \geq \frac{(n-2)^2}{a_2 + \dots + a_{n-1}} = \frac{n-2}{a}.$$

It follows that

$$\begin{aligned} \left(n + \frac{1}{2}\right)^2 &\geq (a_1 + a_2 + \dots + a_n) \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}\right) \\ &\geq (m + (n-2)a + M) \left(\frac{1}{m} + \frac{n-2}{a} + \frac{1}{M}\right) \\ &= (m+M) \left(\frac{1}{m} + \frac{1}{M}\right) + (n-2)^2 + \frac{(n-2)(m+M)}{a} \\ &\quad + (n-2)a \left(\frac{1}{m} + \frac{1}{M}\right) \\ &= \frac{(m+M)^2}{mM} + (n-2)^2 + \frac{(n-2)(m+M)}{mM} \cdot \left(\frac{mM}{a} + a\right) \end{aligned}$$

By the AM-GM Inequality, we have $\frac{mM}{a} + a \geq 2\sqrt{mM}$ with equality at $m \leq a = \sqrt{mM} \leq M$. We deduce that

$$\left(n + \frac{1}{2}\right)^2 \geq \frac{(m+M)^2}{mM} + (n-2)^2 + \frac{2(n-2)(m+M)}{\sqrt{mM}}.$$

Setting $t = \frac{m+M}{\sqrt{mM}}$ in the last inequality yields

$$\left(n + \frac{1}{2}\right)^2 \geq t^2 + (n-2)^2 + 2(n-2)t = (t+n-2)^2,$$

from which it follows that

$$n + \frac{1}{2} \geq t + n - 2.$$

Hence $t \leq 5/2$, which is (1).

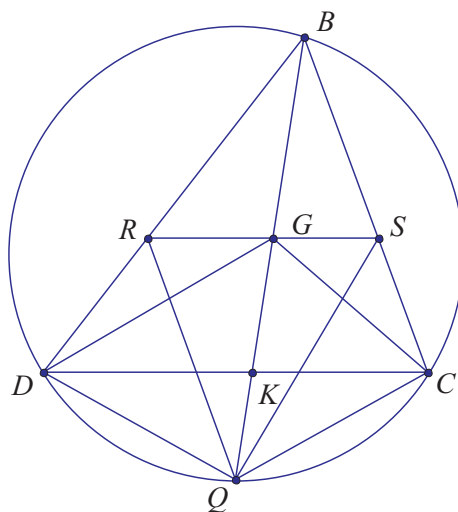
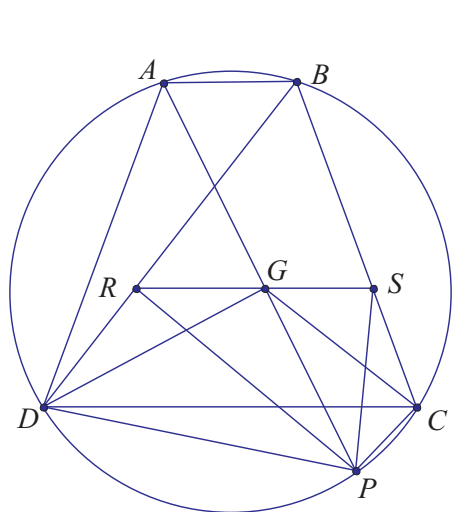
This problem was suggested by Titu Andreescu. The second solution was contributed by Adam Hesterberg, and the third by Zuming Feng.

5. **Solution 1.** First, we prove the “if” part by assuming that ray BG bisects $\angle CBD$; that is, we assume that $\widehat{DQ} = \widehat{CQ}$.

It is easy to see that $ABCD$ is an isosceles trapezoid with $AD = BC$. In particular, $\widehat{AD} = \widehat{BC}$ and $\widehat{AC} = \widehat{BD}$.

Because $ABCPD$ is cyclic, it follows that

$$\angle APC = \frac{\widehat{AC}}{2} = \frac{\widehat{BD}}{2} = \angle BCD = \angle SCD \quad \text{and} \quad \angle APD = \frac{\widehat{AD}}{2} = \frac{\widehat{BC}}{2} = \angle BDC = \angle RDC.$$



Because $RS \parallel DC$, it follows that $180^\circ = \angle GRD + \angle RDC = \angle GRD + \angle APD$ and $180^\circ = \angle GSC + \angle SCD = \angle GSC + \angle APC$; that is, both $GSCP$ and $GRDP$ are cyclic. Hence, $\angle GPR = \angle GDR$ and $\angle GPS = \angle GCS$. In particular, we have

$$\angle RPS = \angle GPR + \angle GPS = \angle GDR + \angle GCS. \tag{3}$$

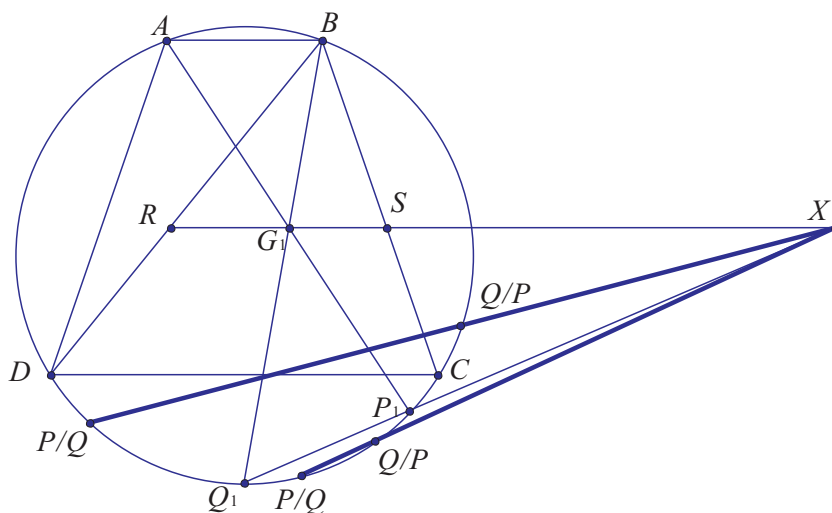
Let K be the intersection of segments BQ and CD . We have $\angle CBK = \angle QBD$ and $\angle KCB = \angle DCB = \angle DQB$; that is, triangles CBK and QBD are similar to each other. Because $RG \parallel CD$, we have $BG/GK = BR/RD$. This means that G and R are the corresponding points in the similar triangles CBK and QBD . Consequently, we have $\angle BCG = \angle BQR$. In exactly the same way, we can show that $\angle BDG = \angle BQS$. Combining the last two equations together with (3) yields

$$\angle RQS = \angle BQS + \angle BQR = \angle BDG + \angle BCG = \angle RDG + \angle SCG = \angle RPS;$$

from which it follows that $PQRS$ is cyclic.

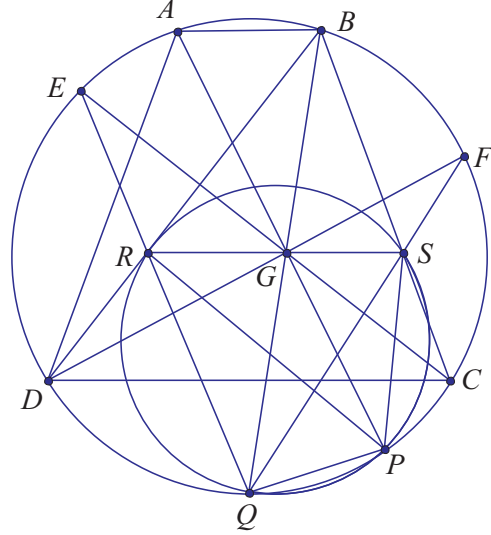
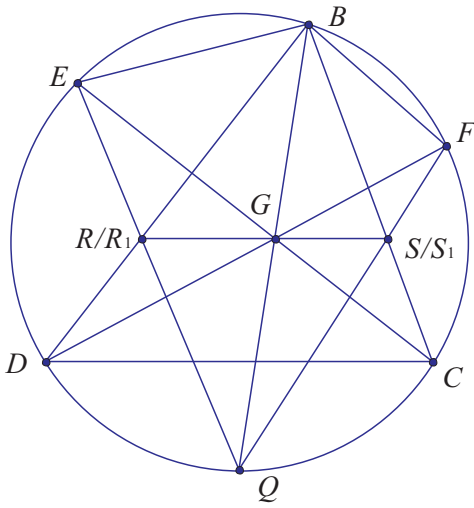
Second, we prove the “only if” part by assuming that $PQRS$ is cyclic. Let γ denote the circumcircle of $PQRS$. We approach indirectly by assuming that ray BG does not bisect $\angle CBD$. Let G_1 be the point on segment RS such that ray BG_1 bisects $\angle CBD$. Let rays AG_1 and BG_1 meet ω again at P_1 and Q_1 (other than A and B). By our proof of the “if” part, P_1Q_1RS is cyclic, and let γ_1 denote its circumcircle.

Hence lines RS, PQ, P_1Q_1 are the radical axes of pairs of circles γ and γ_1 , γ and ω , γ_1 and ω , respectively. Because segments P_1 is the midpoint of arc \widehat{CD} (not including A and B), lines $P_1Q_1 \parallel CD$, implying that lines P_1Q_1 and RS intersect, and let X denote this intersection. Thus X is the radical center of ω, γ, γ_1 . In particular, line PQ also passes through X . We obtain the following configuration.



There are two possibilities for the position of line PQ , namely, (1) both P and Q lie on minor arc $\widehat{P_1Q_1}$; (2) one of P and Q lies on minor arc $\widehat{DQ_1}$ and the other lies on minor arc $\widehat{P_1B}$. If G lies on segment RG_1 , then Q lies on minor arc $\widehat{DQ_1}$, and we must have (2). But in this case, P must lie on minor arc $\widehat{Q_1P_1}$, violating (2). If G lies on segment G_1S , then P must lie on minor arc $\widehat{P_1B}$, and again we must have (2). But in this case, Q must lie on minor arc $\widehat{Q_1C}$, violating (2). In every case, we have a contradiction. Hence our assumption was wrong, and ray BG bisects $\angle CBD$.

Solution 2. We present another approach of the “if” part.



Let rays CG and DG meet ω again at E and F , respectively. Let R_1 denote the intersection of segments BD and QE , and let S_1 denote the intersection of segments BC and QF . Applying Pascal's theorem to cyclic hexagon $BDFQEC$ shows that R_1, G, S_1 are collinear. Because

$$\angle R_1EG = \angle QEC = \frac{\widehat{CQ}}{2} = \frac{\widehat{DQ}}{2} = \angle DBQ = \angle R_1BG,$$

we deduce that $EBGR_1$ is cyclic. Because $EBGR_1$ and $EBCD$ are cyclic, we have

$$\angle BR_1S_1 = \angle BR_1G = \angle BEG = \angle BEC = \angle BDC,$$

from which it follows that $R_1S_1 \parallel CD$; that is, $R_1 = R$ and $S_1 = S$.

Therefore, (3) becomes

$$\angle RPS = \angle GDR + \angle GCS = \angle FDB + \angle BCE = \angle FQB + \angle BQE = \angle FQE = \angle RQS,$$

implying that $PQRS$ is cyclic.

This problem was suggested by Zuming Feng.

6. **Solution 1.** First, we claim there exist i, j such that $(s_i - s_j)(t_i - t_j) \neq 0$. Indeed, for any fixed i , because the sequence s_1, s_2, \dots is nonconstant, there is some j such that $s_j \neq s_i$. If $t_j \neq t_i$ the claim follows, so suppose $t_j = t_i$. Because the sequence t_1, t_2, \dots is nonconstant, there exists k such that $t_k \neq t_i$. If $s_k \neq s_i$ the claim again follows, so suppose $s_k = s_i$. Then $(s_j - s_k)(t_j - t_k) = (s_j - s_i)(t_i - t_k) \neq 0$, and the claim is proven.

We can reorder the pairs (s_i, t_i) relative to each other without affecting either the hypothesis or the conclusion of the problem. So by a suitable reordering, we may assume that $(s_1 - s_2)(t_1 - t_2) \neq 0$.

Second, for any constants a and b , we can replace s_i by $s_i - a$ and t_i by $t_i - b$ for all i without affecting either the hypothesis or the conclusion of the problem (since all the differences $s_i - s_j$ and $t_i - t_j$ remain unchanged). In particular, by taking $a = s_1$ and $b = t_1$, we may assume that $s_1 = t_1 = 0$. So we have reduced the problem to the case $s_1 = t_1 = 0, s_2 \neq 0, t_2 \neq 0$.

Call a pair of positive rational numbers (A, B) *good* if AB is an integer, and As_j and Bt_j are also integers for all j .

Third, we show that a good pair exists.

We know that for all $i \geq 2$, $(s_i - s_1)(t_i - t_1) = s_i t_i$ is an integer; and for all $i, j \geq 2$, $(s_i - s_j)(t_i - t_j) = s_i t_i - s_i t_j - s_j t_i + s_j t_j$ is an integer, which implies $s_i t_j + s_j t_i$ is an integer.

Write the rational numbers s_j, t_j in lowest terms as $s_j = p_j/q_j$ and $t_j = u_j/v_j$. We know that, for each j , $s_j t_j = p_j u_j / q_j v_j$ is an integer. Because u_j is relatively prime to v_j , then, p_j is divisible by v_j , say $p_j = d_j v_j$ for some integer d_j . We also know that

$$s_2 t_j + s_j t_2 = \frac{p_2 u_j}{q_2 v_j} + \frac{p_j u_2}{q_j v_2} = \frac{p_2 u_j q_j v_2 + p_j u_2 q_2 v_j}{q_2 v_j q_j v_2}$$

is an integer. In particular, q_j , being a factor of the denominator, must divide the numerator. But q_j divides $p_2 u_j q_j v_2$, so it also divides the other term, $p_j u_2 q_2 v_j = d_j u_2 q_2 v_j^2$. Since q_j is relatively prime to $p_j = d_j v_j$, it must divide $u_2 q_2$. Moreover, $u_2 q_2 \neq 0$, because of our assumption $t_2 \neq 0$. So we have a positive integer $A = |u_2 q_2|$ such that As_j is an integer for all j . Analogously, we can find a positive integer B such that Bt_j is an integer for all j . This (A, B) constitute a good pair, and existence is proven.

Now we are ready to complete our proof. We know that some good pair exists. We consider a good pair for which the product AB is as small as possible. We will show that $AB = 1$.

Suppose that, for the minimal good pair, $AB > 1$; then AB has a prime factor p . If the integer As_i is divisible by p for all i , then we can divide A by p and obtain a new good pair $(A/p, B)$ having a smaller product than before — a contradiction. So for some i , As_i is not divisible by p . Then Bt_i must be divisible by p , because $s_i t_i$ is an integer and so $(As_i)(Bt_i) = (AB)(s_i t_i)$ is an integer divisible by p . Likewise, there exists some j such that Bt_j is not divisible by p , but As_j is.

Now write

$$(AB)(s_i t_j + s_j t_i) - (As_j)(Bt_i) = (As_i)(Bt_j).$$

All the parenthesized factors are integers, and the left-hand side is divisible by p , but the right-hand side is not. This contradiction completes the proof that the minimal good pair satisfies $AB = 1$.

But now take the minimal good pair (A, B) , and let $r = A$. We have that $s_i r = A s_i$ and $t_i/r = B t_i$ are integers for all i , from which our desired conclusion follows.

Solution 2. For p a prime, define the p -adic norm $\|\cdot\|_p$ on rational numbers as follows: for $r \neq 0$, $\|r\|_p$ is the unique integer n for which we can write $r = p^n a/b$ with a, b integers not divisible by p . (By convention, $\|0\|_p = +\infty$.) We will repeatedly use the well-known (or easy to prove) fact that for any rational numbers r_1, r_2 , we have $\|r_1 \pm r_2\|_p \geq \min(\|r_1\|_p, \|r_2\|_p)$, with equality whenever $\|r_1\|_p \neq \|r_2\|_p$. The condition of the problem implies that

$$\|s_i - s_j\|_p \geq -\|t_i - t_j\|_p \tag{4}$$

for all i, j and all prime p .

We claim in fact that

$$\|s_i - s_j\|_p \geq -\|t_k - t_l\|_p$$

for all i, j, k, l and all prime p . Suppose otherwise; then there exist i, j, k, l, p for which $\|s_i - s_j\|_p < -\|t_k - t_l\|_p$. Since $\|s_i - s_j\|_p = \|(s_i - s_k) - (s_j - s_k)\|_p \geq \min(\|s_i - s_k\|_p, \|s_j - s_k\|_p)$, at least one of $\|s_i - s_k\|_p$ and $\|s_j - s_k\|_p$, say the former, is strictly less than $-\|t_k - t_l\|_p$. By (4), it follows that $\|t_i - t_k\|_p > \|t_k - t_l\|_p$, and thus $\|t_i - t_l\|_p = \|(t_i - t_k) + (t_k - t_l)\|_p = \|t_k - t_l\|_p$. Then by (4) again, $\|s_i - s_l\|_p \geq -\|t_k - t_l\|_p$ and $\|s_k - s_l\|_p \geq -\|t_k - t_l\|_p$, whence $\|s_i - s_k\|_p = \|(s_i - s_l) - (s_k - s_l)\|_p \geq -\|t_k - t_l\|_p$, contradicting the assumption that $\|s_i - s_k\|_p < -\|t_k - t_l\|_p$. This proves the claim.

Now for each prime p , define the integer $f(p) = \min_{i,j} \|s_i - s_j\|_p$. Choose i_0, j_0, k_0, l_0 such that $s_{i_0} \neq s_{j_0}$ and $t_{k_0} \neq t_{l_0}$; then $f(p)$ exists since it is bounded below by $-\|t_{k_0} - t_{l_0}\|_p$ (by the claim) and above by $\|s_{i_0} - s_{j_0}\|_p$. Moreover, if p does not divide the numerator or denominator of either $s_{i_0} - s_{j_0}$ or $t_{k_0} - t_{l_0}$, then $\|s_{i_0} - s_{j_0}\|_p = \|t_{k_0} - t_{l_0}\|_p = 0$ and thus $f(p) = 0$. It follows that $f(p) = 0$ for all but finitely many primes.

We can now define $r = \prod_p p^{-f(p)}$, where the product is over all primes. For any i, j , we have $\|s_i - s_j\|_p \geq f(p)$ for all p by construction, and so $(s_i - s_j)r$ is an integer. On the other hand, for any k, l and any prime p , $\|t_k - t_l\|_p \geq -\|s_i - s_j\|_p$ for all i, j by the claim, and so $\|t_k - t_l\|_p \geq -f(p)$. It follows that $(t_k - t_l)/r$ is an integer for all k, l , whence r is the desired rational number.

This problem and the first solution was suggested by Gabriel Carroll. The second solution was suggested by Lenhard Ng.