38th United States of America Mathematical Olympiad

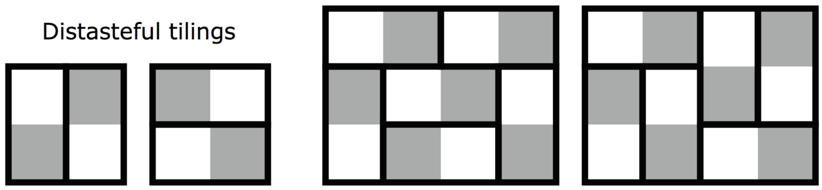
Day I   12:30 PM – 5 PM EDT

April 28, 2009

1. Given circles ω_1 and ω_2 intersecting at points X and Y, let ℓ_1 be a line through the center of ω_1 intersecting ω_2 at points P and Q and let ℓ_2 be a line through the center of ω_2 intersecting ω_1 at points R and S. Prove that if P, Q, R and S lie on a circle then the center of this circle lies on line XY.

2. Let n be a positive integer. Determine the size of the largest subset of $\{-n, -n+1, \ldots, n-1, n\}$ which does not contain three elements a, b, c (not necessarily distinct) satisfying $a + b + c = 0$.

3. We define a *chessboard polygon* to be a polygon whose edges are situated along lines of the form $x = a$ or $y = b$, where a and b are integers. These lines divide the interior into unit squares, which are shaded alternately grey and white so that adjacent squares have different colors. To tile a chessboard polygon by dominoes is to exactly cover the polygon by non-overlapping 1×2 rectangles. Finally, a *tasteful tiling* is one which avoids the two configurations of dominoes shown on the left below. Two tilings of a 3×4 rectangle are shown; the first one is tasteful, while the second is not, due to the vertical dominoes in the upper right corner.



   a) Prove that if a chessboard polygon can be tiled by dominoes, then it can be done so tastefully.

   b) Prove that such a tasteful tiling is unique.