## MOSP 2002 Homework Problems

All students should go to MOSP prepared. The homework problems will be discussed extensively in first few lectures of MOSP. Mathematics Olympiad problems from the world in 2000 and 2001 are selected to make up this homework set. Level I problems are "easier" 2001 problems for all rookies (new MOSP students) to tackle; level II problems are hard 2000 problems for rookies to think about (to develop an idea what to be expected at MOSP) and for vets to review; level III problems are "medium hard" 2001 problems for all vets; level IV problems are "hard" 2001 problems required for team candidates and highly encouraged to all vets. If you have any questions about the homework problems, please contact either Zuming Feng, Phillips Exeter Academy, 20 Main St., NH 03833, 603-777-4368, zfeng@exeter.edu or George Lee, 1822 Harvard Yard Mail Center, Cambridge, MA 02138, 617-493-5923, lee43@fas.harvard.edu.

# Part I

1. Find all triples $a, b, c$ of real numbers for which a real number $x$ satisfies

$$\sqrt{2x^2 + ax + b} > x - c$$

   if and only if $x \leq 0$ or $x > 1$.

2. In a certain language there are $n$ letters. A sequence of letters is called a *word* if and only if between any pair of identical letters, there is no other pair of equal letters. Prove that there exists a word of maximum possible length, and find the number of words which have that length.

3. Every vertex of the unit squares on an $m \times n$ chessboard is colored either blue, green, or red, such that all the vertices on the boundary of the board are colored red. We say that a unit square of the board is *properly colored* if exactly one pair of adjacent vertices of the square are the same color. Show that the number of properly colored squares is even.

4. Given an odd prime $p$, find all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying the following two conditions:

   (i) $f(m) = f(n)$ for all $m, n \in \mathbb{Z}$ such that $m \equiv n \pmod{p}$;

   (ii) $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{Z}$.

5. Let $n \geq 2$ be an integer. Show that

$$\sum_{k=1}^{n} k x_k \leq \binom{n}{2} + \sum_{k=1}^{n} x_k^k$$

   for all nonnegative reals $x_1, x_2, \ldots, x_n$.

6. We flip a fair coin repeatedly until encountering three consecutive flips of the form (i) two tails followed by heads, or (ii) heads, followed by tails, followed by heads. Which sequence, (i) or (ii), is more likely to occur first?

7. Let a *generalized diagonal* in an $n \times n$ matrix be a set of entries which contains exactly one element from each row and one element from each column. Let $A$ be an $n \times n$ matrix filled with 0s and 1s which contains exactly one generalized diagonal whose entries are all 1. Prove that it is possible to permute the rows and columns of $A$ to obtain an *upper-triangular matrix*, a matrix $(b_{ij})_{1\le i,j\le n}$ such that $b_{ij} = 0$ whenever $1 \le j < i \le n$.

8. Peter and Alex play a game starting with an ordered pair of integers $(a, b)$. On each turn, the current player increases or decreases either $a$ or $b$: Peter by 1, and Alex by 1 or 3. Alex wins if at some point in the game the roots of $x^2 + ax + b$ are integers. Is it true that given any initial values $a$ and $b$, Alex can guarantee that he wins?

9. Let $M$ and $N$ be points on sides $\overline{AB}$ and $\overline{BC}$, respectively, of parallelogram $ABCD$ such that $AM = NC$. Let $Q$ be the intersection of $\overline{AN}$ and $\overline{CM}$. Prove that $\overline{DQ}$ is an angle bisector of angle $CDA$.

10. A target consists of an equilateral triangle broken into 100 equilateral triangles of unit side length by three sets of parallel lines. A sniper shoots at the target repeatedly as follows: he aims at one of the small triangles and then hits either that triangle or one of the small triangles which shares a side with it. He may choose to stop shooting at any time. What is the greatest number of triangles that he can be sure to hit exactly five times?

11. A circle is tangent to rays $OA$ and $OB$ at $A$ and $B$, respectively. Let $K$ be a point on minor arc $AB$ of this circle. Let $L$ be a point on line $OB$ such that $\overline{OA} \parallel \overline{KL}$. Let $M$ be the intersection (distinct from $K$) of line $AK$ and the circumcircle $\omega$ of triangle $KLB$. Prove that line $OM$ is tangent to $\omega$.

12. Let $a_1, a_2, \ldots, a_{10^6}$ be nonzero integers between 1 and 9, inclusive. Prove that at most 100 of the numbers $\overline{a_1 a_2 \ldots a_k}$ $(1 \le k \le 10^6)$ are perfect squares.

## Part II

13. The feet of the angle bisectors of triangle $ABC$ are $X$, $Y$, and $Z$. The circumcircle of triangle $XYZ$ cuts off three segments from lines $AB$, $BC$, and $CA$. Prove that two of these segments' lengths add up to the third segment's length.

14. In the plane the triangle $A_0B_0C_0$ is given. Consider all triangles $ABC$ satisfying the following conditions: (i) $C_0$, $A_0$, and $B_0$ lie on $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, respectively; (ii) $\angle ABC = \angle A_0B_0C_0$, $\angle BCA = \angle B_0C_0A_0$, and $\angle CAB = \angle C_0A_0B_0$. Find the locus of the circumcenter of all such triangles $ABC$.

15. Let $M = \{1, 2, \ldots, 40\}$. Find the smallest positive integer $n$ for which it is possible to partition $M$ into $n$ disjoint subsets such that whenever $a$, $b$, and $c$ (not necessarily distinct) are in the same subset, $a \ne b + c$.

16. We are given distinct positive integers $a_1, a_2, \ldots, a_{20}$. The set of pairwise sums $\{a_i + a_j \mid 1 \le i \le j \le 20\}$ contains 201 elements. What is the smallest possible number of elements in the set $\{|a_i - a_j| \mid 1 \le i < j \le 20\}$, the set of positive differences between the integers?

17. A table tennis club wishes to organize a doubles tournament, a series of matches where in each match one pair of players competes against a pair of two different players. Let a player's *match number* for a tournament be the number of matches he or she participates in. We are given a set $A = \{a_1, a_2, \ldots, a_k\}$ of distinct positive integers all divisible by 6. Find with proof the minimal number of players among whom we can schedule a doubles tournament such that

   (i) each participant belongs to at most 2 pairs;

   (ii) any two different pairs have at most 1 match against each other;

   (iii) if two participants belong to the same pair, they never compete against each other; and

   (iv) the set of the participants' match numbers is exactly $A$.

18. There are 2000 cities in a country, and each pair of cities is connected by either no roads or exactly one road. A *cyclic path* is a nonempty, connected path of roads such that each city is at the end of either 0 or 2 roads in the path. For every city, there are at most $N$ cyclic paths which both pass through this city and contain an odd number of roads. Prove that the country can be separated into $2N+2$ republics such that any two cities from the same republic are not connected by a road.

19. Let $ABC$ be a nonequilateral triangle. Suppose there is an interior point $P$ such that the three cevians through $P$ all have the same length $\lambda$ where $\lambda < \min\{AB, BC, CA\}$. Show that there is another interior point $P' \neq P$ such that the three cevians through $P'$ also are of equal length.

20. Isosceles triangles $A_3A_1O_2$ and $A_1A_2O_3$ are constructed externally along the sides of a triangle $A_1A_2A_3$ with $O_2A_3 = O_2A_1$ and $O_3A_1 = O_3A_2$. Let $O_1$ be a point on the opposite side of line $A_2A_3$ as $A_1$ with $\angle O_1A_3A_2 = \frac{1}{2}\angle A_1O_3A_2$ and $\angle O_1A_2A_3 = \frac{1}{2}\angle A_1O_2A_3$, and let $T$ be the foot of the perpendicular from $O_1$ to $\overline{A_2A_3}$. Prove that $\overline{A_1O_1} \perp \overline{O_2O_3}$ and that $\frac{A_1O_1}{O_2O_3} = 2\frac{O_1T}{A_2A_3}$.

21. Given a permutation $(a_0, a_1, \ldots, a_n)$ of the sequence $0, 1, \ldots, n$, a transposition of $a_i$ with $a_j$ is called *legal* if $a_i = 0$, $i > 0$, and $a_{i-1} + 1 = a_j$. The permutation $(a_0, a_1, \ldots, a_n)$ is called *regular* if after finitely many legal transpositions it becomes $(1, 2, \ldots, n, 0)$. For which numbers $n$ is the permutation $(1, n, n-1, \ldots, 3, 2, 0)$ regular?

22. Is it possible to select 102 17-element subsets of a 102-element set, such that the intersection of any two of the subsets has at most 3 elements?

23. Suppose that $a, b, c$ are real numbers such that for any positive real numbers $x_1, x_2, \ldots, x_n$, we have

$$\left(\frac{\sum_{i=1}^n x_i}{n}\right)^a \cdot \left(\frac{\sum_{i=1}^n x_i^2}{n}\right)^b \cdot \left(\frac{\sum_{i=1}^n x_i^3}{n}\right)^c \geq 1.$$

   Prove that the vector $(a, b, c)$ has the form $p(-2, 1, 0) + q(1, -2, 1)$ for some nonnegative real numbers $p$ and $q$.

24. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$|f(x+y) - f(x) - f(y)| \leq 1$$

   for all $x, y \in \mathbb{R}$. Show that there exists a function $g : \mathbb{R} \to \mathbb{R}$ with $|f(x) - g(x)| \leq 1$ for all $x \in \mathbb{R}$, and with $g(x+y) = g(x) + g(y)$ for all $x, y \in \mathbb{R}$.

## Part III

25. The convex quadrilateral $ABCD$ is inscribed in the circle $S_1$. Let $O$ be the intersection of $\overline{AC}$ and $\overline{BD}$. Circle $S_2$ passes through $D$ and $O$, intersecting $\overline{AD}$ and $\overline{CD}$ at $M$ and $N$, respectively. Lines $OM$ and $AB$ intersect at $R$, lines $ON$ and $BC$ intersect at $T$, and $R$ and $T$ lie on the same side of line $BD$ as $A$. Prove that $O$, $R$, $T$, and $B$ are concyclic.

26. There are $n$ aborigines on an island. Any two of them are either friends or enemies. One day, the chieftain orders that all citizens (including himself) make and wear a necklace with zero or more stones so that (i) given a pair of friends, there exists a color such that each has a stone of that color; (ii) given a pair of enemies, there does *not* exist a color such that each a stone of that color.

    (a) Prove that the aborigines can carry out the chieftain's order.

    (b) What is the minimum number of colors of stones required for the aborigines to carry out the chieftain's order?

27. Three real numbers $a, b, c \geq 0$ satisfy the inequalities $a^2 \leq b^2 + c^2$, $b^2 \leq c^2 + a^2$, and $c^2 \leq a^2 + b^2$. Prove that

$$(a + b + c)(a^2 + b^2 + c^2)(a^3 + b^3 + c^3) \geq 4(a^6 + b^6 + c^6),$$

and determine when equality holds.

28. Let $p$ be a prime number and $m$ be a positive integer. Show that there exists a positive integer $n$ such that there exist $m$ consecutive zeroes in the decimal representation of $p^n$.

29. In triangle $ABC$, $\angle ABC < \pi/4$. Point $D$ lies on $\overline{BC}$ so that the incenter of triangle $ABD$ coincides with the circumcenter $O$ of triangle $ABC$. Let $\omega$ be the circumcenter of triangle $AOC$. Let $P$ be the point of intersection of the two tangent lines to $\omega$ at $A$ and $C$. Let $Q$ be the point of intersection of lines $AD$ and $CO$, and let $X$ be the point of intersection of line $PQ$ and the tangent line to $\omega$ at $O$. Show that $XO = XD$.

    Consider the set $A$ of all positive integers $n$ with the following properties: the decimal expansion contains no 0, and the sum of the (decimal) digits of $n$ divides $n$.

    (a) Prove that there exist infinitely many elements in $A$ with the following properties: the digits that appear in the decimal expansion of $A$ appear the same number of times.

    (b) Show that for each positive integer $k$, there exists an element in $A$ with exactly $k$ digits.

30. Let $O$ and $H$ be the circumcenter and orthocenter, respectively, of triangle $ABC$. The *nine-point circle* of triangle $ABC$ is the circle passing through the midpoints of the sides, the feet of the altitudes, and the midpoints of $\overline{AH}$, $\overline{BH}$, and $\overline{CH}$. Let $N$ be the center of this circle, and let $N'$ be the point such that

$$\angle N'BA = \angle NBC \qquad \text{and} \qquad \angle N'AB = \angle NAC.$$

Let the perpendicular bisector of $\overline{OA}$ intersect line $BC$ at $A'$, and define $B'$ and $C'$ similarly. Prove that $A'$, $B'$, and $C'$ lie on a line $\ell$ which is perpendicular to line $ON'$.

31. (a) Let $f$, $g$: $\mathbb{Z} \to \mathbb{Z}$ be injective maps. Show that the function $h : \mathbb{Z} \to \mathbb{Z}$, defined by $h(x) = f(x)g(x)$ for all $x \in \mathbb{Z}$, cannot be surjective.

(b) Let $f$: $\mathbb{Z} \to \mathbb{Z}$ be a surjective map. Show that there exist surjective functions $g, h$: $\mathbb{Z} \to \mathbb{Z}$ such that $f(x) = g(x)h(x)$ for all $x \in \mathbb{Z}$.

32. Three schools each have 200 students. Every student has at least one friend in each school, where if student $a$ is a friend of student $b$, then $b$ is a friend of $a$. There exists a set $E$ of 300 students (chosen from among the 600 students at the three schools) with the following property: for any school $S$ and any two students $x, y \in E$ who are not in the school $S$, $x$ and $y$ do not have the same number of friends in $S$. Show that one can find three students, one in each school, such that any two are friends with each other.

33. The polynomial $P(x) = x^3 + ax^2 + bx + c$ has three distinct real roots. The polynomial $P(Q(x))$, where $Q(x) = x^2 + x + 2001$, has no real roots. Prove that $P(2001) > \frac{1}{64}$.

34. Each number $1, 2, \ldots, n^2$ is written once in an $n \times n$ grid such that each square contains one number. Given any two squares in the grid, a vector is drawn from the center of the square containing the larger number to the center of the other square. If the sums of the numbers in each row or column of the grid are equal, prove that the sum of the drawn vectors is zero.

35. Points $A_1$, $B_1$, $C_1$ are selected inside triangle $ABC$ on the altitudes from $A$, $B$, and $C$, respectively. If $[ABC_1]+[BCA_1]+[CAB_1] = [ABC]$, prove that the circumcircle of triangle $A_1B_1C_1$ passes through $H$.

36. A $3 \times 3 \times 3$ cube is divided into 27 congruent $1 \times 1 \times 1$ cells. One of these cells is empty, and the others are filled with unit cubes labelled $1, 2, \ldots, 26$ in some order. An *admissible move* consists of moving a unit cube which shares a face with the empty cell into the empty cell. Does there always exist — for any initial empty cell and any labelling of the 26 cubes — a finite sequence of admissible moves after which each unit cube labelled with $k$ is in the cell originally containing the unit cube labelled with $27 - k$, for each $k = 1, 2, \ldots, 26$?

## Part IV

37. Let $ABCD$ be a rectangle, and let $\Gamma$ be an arc of a circle passing through $A$ and $C$. Let $\Gamma_1$ be a circle which is tangent to lines $CD$ and $DA$ as well as tangent to $\Gamma$. Similarly, let $\Gamma_2$ be a circle lying completely inside rectangle $ABCD$ which is tangent to lines $AB$ and $BC$ as well as tangent to $\Gamma$. Suppose that $\Gamma_1$ and $\Gamma_2$ both lie completely in the closed region bounded by rectangle $ABCD$. Let $r_1$ and $r_2$ be the radii of $\Gamma_1$ and $\Gamma_2$, respectively, and let $r$ be the inradius of triangle $ABC$.

(a) Prove that $r_1 + r_2 = 2r$.

(b) Show that one of the common internal tangents to $\Gamma_1$ and $\Gamma_2$ is parallel to $\overline{AC}$ and has length $|AB - BC|$.

38. The circles $k_1$ and $k_2$ and the point $P$ lie in a plane. There exists a line $\ell$ and points $A_1, A_2, B_1, B_2, C_1, C_2$ with the following properties: $\ell$ passes through $P$ and intersects $k_i$ at $A_i$ and $B_i$ for $i = 1, 2$; $C_i$ lies

on $k_i$ for $i = 1, 2$; and $A_1C_1 = B_1C_1 = A_2C_2 = B_2C_2$. Describe how to construct such a line and such points given only $k_1$, $k_2$, and $P$.

39. The set of *n-variable formulas* is a subset of the functions of $n$ variables $x_1, \ldots, x_n$, and it is defined recursively as follows: the formulas $x_1, \ldots, x_n$ are $n$-variable formulas, as is any formula of the form

$$(x_1, \ldots, x_n) \mapsto \max\{f_1(x_1, \ldots, x_n), \ldots, f_k(x_1, \ldots, x_n)\}$$

or

$$(x_1, \ldots, x_n) \mapsto \min\{f_1(x_1, \ldots, x_n), \ldots, f_k(x_1, \ldots, x_n)\},$$

where each $f_i$ is an $n$-variable formula. For example,

$$\max(x_2, x_3, \min(x_1, \max(x_4, x_5)))$$

is a 5-variable formula. Suppose that $P$ and $Q$ are two $n$-variable formulas such that

$$P(x_1, \ldots, x_n) = Q(x_1, \ldots, x_n) \qquad (*)$$

for all $x_1, \ldots, x_n \in \{0, 1\}$. Prove that $(*)$ also holds for all $x_1, \ldots, x_n \in \mathbb{R}$.

40. Let $n \geq 5$ be a positive integer, and let $a_1$, $b_1$, $a_2$, $b_2$, $\ldots$, $a_n$, $b_n$ be integers satisfying the following two conditions:

    (i) the pairs $(a_i, b_i)$ are all distinct for $i = 1, 2, \ldots, n$;

    (ii) $|a_i b_{i+1} - a_{i+1} b_i| = 1$ for $i = 1, 2, \ldots, n$, where $(a_{n+1}, b_{n+1}) = (a_1, b_1)$.

    Show that there exist $i, j$ with $1 \leq i, j \leq n$ such that $1 < |i - j| < n - 1$ and $|a_i b_j - a_j b_i| = 1$.

41. Let $n_1 < n_2 < \cdots < n_{2000} < 10^{100}$ be positive integers. Prove that one can find two nonempty disjoint subsets $A$ and $B$ of $\{n_1, n_2, \ldots, n_{2000}\}$ such that $|A| = |B|$, $\sum_{x \in A} x = \sum_{x \in B} x$, and $\sum_{x \in A} x^2 = \sum_{x \in B} x^2$.

42. Prove that there is no function $f : (0, \infty) \to (0, \infty)$ such that

$$f(x + y) \geq f(x) + y f(f(x))$$

for all $x, y \in (0, \infty)$.

43. Let $P$ be a convex polyhedron with vertices $V_1, V_2, \ldots, V_p$. Two vertices $V_i$ and $V_j$ are called *neighbors* if they are distinct and belong to the same face of the polyhedron. The $p$ sequences $(v_i(n))_{n \geq 0}$, for $i = 1, 2, \ldots, p$, are defined recursively as follows: the $v_i(0)$ are chosen arbitrarily; and for $n \geq 0$, $v_i(n + 1)$ is the arithmetic mean of the numbers $v_j(n)$ for all $j$ such that $V_i$ and $V_j$ are neighbors. Suppose that $v_i(n)$ is an integer for all $1 \leq i \leq p$ and $n \in \mathbb{N}$. Prove that there exist $N \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that $v_i(n) = k$ for all $n \geq N$ and $i = 1, 2, \ldots, p$.

44. Let $a$ and $b$ be distinct positive integers such that $ab(a + b)$ is divisible by $a^2 + ab + b^2$. Prove that $|a - b| > \sqrt[3]{ab}$.

45. Let $SABC$ be a tetrahedron. The circumcircle of $ABC$ is a great circle of a sphere $\omega$, and $\omega$ intersects $\overline{SA}$, $\overline{SB}$, and $\overline{SC}$ again at $A_1$, $B_1$, and $C_1$, respectively. The planes tangent to $\omega$ at $A_1$, $B_1$, and $C_1$ intersect at a point $O$. Prove that $O$ is the circumcenter of tetrahedron $SA_1B_1C_1$.

46. The sequence $a_1, a_2, \ldots, a_{2010}$ has the following properties:

- any 20 consecutive terms of the sequence have nonnegative sum;
- $|a_i a_{i+1}| \leq 1$ for $i = 1, 2, \ldots, 2009$.

Determine the maximum possible value of $\sum_{i=1}^{2010} a_i$.

47. Two triangles $ABC$ and $PQR$ satisfy the following properties: $A$ and $P$ are the midpoints of $\overline{QR}$ and $\overline{BC}$, respectively, and lines $QR$ and $BC$ are the bisectors of angles $BAC$ and $QPR$, respectively. Prove that $AB + AC = PQ + PR$.

48. Let $I$ and $I_a$ be the incenter and excenter opposite $A$, respectively, of triangle $ABC$. Suppose that $\overline{II_a}$ meets $\overline{BC}$ and the circumcircle of triangle $ABC$ at $A'$ and $M$, respectively. Let $N$ be the midpoint of arc $MBA$ of the circumcircle of triangle $ABC$. Let lines $NI$ and $NI_a$ intersect the incircle of triangle $ABC$ at $S$ and $T$, respectively. Prove that $S$, $T$, and $A'$ are collinear.