XXII. Student Worksheet

We present four snapshots (ideas) contained in problems from the 2002 AMC 10/12 contests, one from each exam (10A, 10B, 12A, 12B), along with another problem based on each idea. Answers to the challenge problems can be found on page 2 of this manual: Changes and Important Procedures.

AMC 10A, Problem #16 – "Sum Them All"

- If $a + 1 = b + 2 = c + 3 = d + 4 = a + b + c + d + 5$, then $a + b + c + d$ is

  (A) -5 (B) -10/3 (C) -7/3 (D) 5/3 (E) 5

- **Solution (B)** Because a, b, c, d appear exactly once in each of the left-hand side of equalities:

$$a + 1 = a + b + c + d + 5$$
$$b + 2 = a + b + c + d + 5$$
$$c + 3 = a + b + c + d + 5$$
$$d + 4 = a + b + c + d + 5$$

  sum up all these relations and get

$$(a + b + c + d) + 10 = 4(a + b + c + d) + 20.$$

  We obtain $a + b + c + d = -10/3$

You can use the same method to solve problem 19 on the 2002 AMC 10P:

- If a, b, c are real numbers such that $a^2 + 2b = 7$, $b^2 + 4c = -7$, and $c^2 + 6a = -14$, find $a^2 + b^2 + c^2$.

  (A) 14 (B) 21 (C) 28 (D) 35 (E) 49

AMC 10B, Problem #20 – "Use the Symmetry"

- Let a, b, and c be real numbers such that $a - 7b + 8c = 4$ and $8a + 4b - c = 7$. Then $a^2 - b^2 + c^2$ is

  (A) 0 (B) 1 (C) 4 (D) 7 (E) 8

- **Solution (B)** Exploit the symmetry with respect to the absolute values of the coefficients to a and c and with respect to the coefficient to b and the free term, respectively, by writing the given relations as $a + 8c = 4 + 7b$ and $8a - c = 7 - 4b$. Squaring both equations and adding up the results yields

$$(a + 8c)^2 + (8a - c)^2 = (4 + 7b)^2 + (7 - 4b)^2.$$

Expanding gives $65(a^2 + c^2) = 65(1 + b^2)$. So $a^2 + c^2 = 1 + b^2$, and $a^2 - b^2 + c^2 = 1$.

You can use the same technique to solve problem 27 on the 1999 AHSME:

- In triangle ABC, $3 \sin A + 4 \cos B = 6$ and $4 \sin B + 3 \cos A = 1$. Then $\angle C$ in degrees is:

  (A) 30 (B) 60 (C) 90 (D) 120 (E) 150

Answers for the challenge problems are listed at the bottom of page 2 of this Manual: Changes and Important Procedures

AMC 12A, Problem #24 - "Take the module"

- Find the number of ordered pairs of real numbers (a, b) such that $(a + bi)^{2002} = a - bi$.

  (A) 1001 (B) 1002 (C) 2001 (D) 2002 (E) 2004

- **Solution (E)** Let $z = a + bi, \bar{z} = a - bi$, and $|z| = \sqrt{a^2 + b^2}$. The given relation becomes $z^{2002} = \bar{z}$. Note that

$$|z|^{2002} = |z^{2002}| = |\bar{z}| = |z|,$$

from which it follows that

$$|z|(|z|^{2001} - 1) = 0.$$

Hence $|z| = 0$, and $(a, b) = (0, 0)$, or $|z| = 1$. In the case $|z| = 1$, we have $z^{2002} = \bar{z}$, which is equivalent to $z^{2003} = \bar{z} \cdot z = |z|^2 = 1$. Since the equation $z^{2003} = 1$ has 2003 distinct solutions, there are altogether $1 + 2003 = 2004$ ordered pairs that meet the required conditions.

Use the same idea to solve the following 1999 make-up AIME problem:

- Find the number of complex numbers z such that

$$z^{19} = (\bar{z})^{99}$$

where $\bar{z}$ is the conjugate of z.

AMC 12B, Problem #24 – "Employ the concept of area (or volume)"

- A convex quadrilateral $ABCD$ with area 2002 contains a point P in its interior such that $PA = 24$, $PB = 32$, $PC = 28$ and $PD = 45$. Find the perimeter of $ABCD$.

  (A) $4\sqrt{2002}$ (B) $2\sqrt{8465}$ (C) $2(48 + \sqrt{2002})$ (D) $2\sqrt{8633}$ (E) $4(36 + \sqrt{113})$

- **Solution (E)** We have

$$\text{Area}(ABCD) \le \frac{1}{2} AC \cdot BD,$$

with equality if and only if $AC \perp BD$. Since

$$2002 = \text{Area}(ABCD) \le \frac{1}{2} AC \cdot BD \le \frac{1}{2}(AP + PC) \cdot (BP + PD) = \frac{52 \cdot 77}{2} = 2002,$$

it follows that the diagonals AC and BD are perpendicular and intersect at P. Thus, $AB = \sqrt{24^2 + 32^2} = 40$, $BC = \sqrt{28^2 + 32^2} = 4\sqrt{113}$, $CD = \sqrt{28^2 + 45^2} = 53$, and $DA = \sqrt{45^2 + 24^2} = 51$. The perimeter of $ABCD$ is therefore

$$144 + 4\sqrt{113} = 4\left(36 + \sqrt{113}\right)$$

Using the idea of writing the volume of a tetrahedron turns useful in solving problem 24 on the 2002 AMC 12P:

- Let $ABCD$ be a regular tetrahedron and let E be a point inside the face ABC. Denote by s the sum of the distances from E to the faces DAB, DBC, DCA, and by S the sum of the distances from E to the edges AB, BC CA. Then $\frac{s}{S}$ equals

  (A) $\sqrt{2}$ (B) $\frac{2\sqrt{2}}{3}$ (C) $\frac{\sqrt{6}}{2}$ (D) 2 (E) 3