

33rd United States of America Mathematical Olympiad

Day I 12:30 PM – 5 PM EDT

April 27, 2004

1. Let $ABCD$ be a quadrilateral circumscribed about a circle, whose interior and exterior angles are at least 60° . Prove that

$$\frac{1}{3}|AB^3 - AD^3| \leq |BC^3 - CD^3| \leq 3|AB^3 - AD^3|.$$

When does equality hold?

2. Suppose a_1, \dots, a_n are integers whose greatest common divisor is 1. Let S be a set of integers with the following properties.
- (a) For $i = 1, \dots, n$, $a_i \in S$.
 - (b) For $i, j = 1, \dots, n$ (not necessarily distinct), $a_i - a_j \in S$.
 - (c) For any integers $x, y \in S$, if $x + y \in S$, then $x - y \in S$.

Prove that S must be equal to the set of all integers.

3. For what real values of $k > 0$ is it possible to dissect a $1 \times k$ rectangle into two similar, but noncongruent, polygons?