

38th United States of America Mathematical Olympiad

Day II 12:30 PM – 5 PM EDT

April 29, 2009

4. For $n \geq 2$ let a_1, a_2, \dots, a_n be positive real numbers such that

$$(a_1 + a_2 + \dots + a_n) \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right) \leq \left(n + \frac{1}{2} \right)^2.$$

Prove that $\max(a_1, a_2, \dots, a_n) \leq 4 \min(a_1, a_2, \dots, a_n)$.

5. Trapezoid $ABCD$, with $\overline{AB} \parallel \overline{CD}$, is inscribed in circle ω and point G lies inside triangle BCD . Rays AG and BG meet ω again at points P and Q , respectively. Let the line through G parallel to \overline{AB} intersect \overline{BD} and \overline{BC} at points R and S , respectively. Prove that quadrilateral $PQRS$ is cyclic if and only if \overline{BG} bisects $\angle CBD$.
6. Let s_1, s_2, s_3, \dots be an infinite, nonconstant sequence of rational numbers, meaning it is not the case that $s_1 = s_2 = s_3 = \dots$. Suppose that t_1, t_2, t_3, \dots is also an infinite, nonconstant sequence of rational numbers with the property that $(s_i - s_j)(t_i - t_j)$ is an integer for all i and j . Prove that there exists a rational number r such that $(s_i - s_j)r$ and $(t_i - t_j)/r$ are integers for all i and j .