- Let $P(x) = (x-1)(x-2)(x-3)$. For how many polynomials $Q(x)$ does there exist a polynomial $R(x)$ of degree 3 such that $P(Q(x)) = P(x) \cdot R(x)$?

(A) 19 **(B)** 22 **(C)** 24 **(D)** 27 **(E)** 32

2005 AMC 12 A, Problem #24— "What degree does $P(x) \cdot R(x)$ have?"

- **Solution (B)** The polynomial $P(x) \cdot R(x)$ has degree 6, so $Q(x)$ must have degree 2. Therefore Q is uniquely determined by the ordered triple $(Q(1), Q(2), Q(3))$. When $x = 1$, 2, or 3, we have $0 = P(x) \cdot R(x) = P(Q(x))$. It follows that $(Q(1), Q(2), Q(3))$ is one of the 27 ordered triples (i, j, k), where i, j, and k can be chosen from the set $\{1, 2, 3\}$. However, the choices $(1, 1, 1)$, $(2, 2, 2)$, $(3, 3, 3)$, $(1, 2, 3)$, and $(3, 2, 1)$ lead to polynomials $Q(x)$ defined by $Q(x) = 1$, 2, 3, x, and $4 - x$, respectively, all of which have degree less than 2. The other 22 choices for $(Q(1), Q(2), Q(3))$ yield non-collinear points, so in each case $Q(x)$ is a quadratic polynomial.

Difficulty: Hard

NCTM Standard: Algebra Standard: Analyze change in various contexts

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