- Let S be the set of all points with coordinates (x, y, z), where x, y, and z are each chosen from the set $\{0, 1, 2\}$. How many equilateral triangles have all their vertices in S ?

(A) 72 **(B)** 76 **(C)** 80 **(D)** 84 **(E)** 88

2005 AMC 12 A, Problem #25— "Good Luck"

- **Solution (C)** Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, and $C(x_3, y_3, z_3)$ be the vertices of such a triangle. Let

$$(\Delta x_k, \Delta y_k, \Delta z_k) = (x_{k+1} - x_k, y_{k+1} - y_k, z_{k+1} - z_k), \text{ for } 1 \le k \le 3,$$

where $(x_4, y_4, z_4) = (x_1, y_1, z_1)$. Then $(|\Delta x_k|, |\Delta y_k|, |\Delta z_k|)$ is a permutation of one of the ordered triples $(0, 0, 1)$, $(0, 0, 2)$, $(0, 1, 1)$, $(0, 1, 2)$, $(0, 2, 2)$, $(1, 1, 1)$, $(1, 1, 2)$, $(1, 2, 2)$, or $(2, 2, 2)$. Since $\triangle ABC$ is equilateral, $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ correspond to permutations of the same ordered triple (a, b, c). Because

$$\sum_{k=1}^{3} \Delta x_k = \sum_{k=1}^{3} \Delta y_k = \sum_{k=1}^{3} \Delta z_k = 0,$$

the sums

$$\sum_{k=1}^{3} |\Delta x_k|, \quad \sum_{k=1}^{3} |\Delta y_k|, \quad \text{and} \quad \sum_{k=1}^{3} |\Delta z_k|$$

are all even. Therefore $(|\Delta x_k|, |\Delta y_k|, |\Delta z_k|)$ is a permutation of one of the triples $(0, 0, 2)$, $(0, 1, 1)$, $(0, 2, 2)$, $(1, 1, 2)$, or $(2, 2, 2)$.

If $(a, b, c) = (0, 0, 2)$, each side of $\triangle ABC$ is parallel to one of the coordinate axes, which is impossible.

If $(a, b, c) = (2, 2, 2)$, each side of $\triangle ABC$ is an interior diagonal of the $2 \times 2 \times 2$ cube that contains S, which is also impossible.

If $(a, b, c) = (0, 2, 2)$, each side of $\triangle ABC$ is a face diagonal of the $2 \times 2 \times 2$ cube that contains S. The three faces that join at any vertex determine such a triangle, so the triple $(0, 2, 2)$ produces a total of 8 triangles.

If $(a, b, c) = (0, 1, 1)$, each side of $\triangle ABC$ is a face diagonal of a unit cube within the larger cube that contains S. There are 8 such unit cubes producing a total of $8 \cdot 8 = 64$ triangles.

There are two types of line segments for which $(a, b, c) = (1, 1, 2)$. One type joins the center of the face of the $2 \times 2 \times 2$ cube to a vertex on the opposite face. The other type joins the midpoint of one edge of the cube to the midpoint of another edge. Only the second type of segment can be a side of $\triangle ABC$. The midpoint of each of the 12 edges is a vertex of two suitable triangles, so there are $12 \cdot 2/3 = 8$ such triangles.

The total number of triangles is $8 + 64 + 8 = 80$.

Difficulty: Hard

NCTM Standard: Geometry Standard: Specify locations and describe spatial relationships using coordinate geometry and other representational systems

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