How many pairs of positive integers (a, b) are there such that a and b have no common factors greater than 1 and

$$\frac{a}{b} + \frac{14b}{9a}$$

is an integer?

(A) 4 **(B)** 6 **(C)** 9 **(D)** 12 **(E)** infinitely many

2007 AMC 10 B, Problem #25—
2007 AMC 12 B, Problem #24—
"Play with the restriction: $\frac{a}{b} + \frac{14b}{9a}$ is an integer."

Solution

Answer (A): Let $u = a/b$. Then the problem is equivalent to finding all positive rational numbers u such that $u + \frac{14}{9u} = k$, for some integer k. This equation is equivalent to $9u^2 - 9uk + 14 = 0$, whose solutions are

$$u = \frac{9k \pm \sqrt{81k^2 - 504}}{18} = \frac{k}{2} \pm \frac{1}{6}\sqrt{9k^2 - 56}.$$

Hence u is rational if and only if $\sqrt{9k^2 - 56}$ is rational, which is true if and only if $9k^2 - 56$ is a perfect square. Suppose that $9k^2 - 56 = s^2$ for some positive integer s. Then $(3k - s)(3k + s) = 56$. The only factors of 56 are 1, 2, 4, 7, 8, 14, 28, and 56, so $(3k - s, 3k + s)$ is one of the ordered pairs $(1, 56)$, $(2, 28)$, $(4, 14)$, or $(7, 8)$. The cases $(1, 56)$ and $(7, 8)$ yield no integer solutions. The cases $(2, 28)$ and $(4, 14)$ yield $k = 5$ and $k = 3$, respectively. If $k = 5$, then $u = 1/3$ or $u = 14/3$. If $k = 3$, then $u = 2/3$ or $u = 7/3$. Therefore there are four pairs (a, b) that satisfy the given conditions, namely $(1, 3)$, $(2, 3)$, $(7, 3)$, and $(14, 3)$.

OR

Rewrite the equation $\dfrac{a}{b} + \dfrac{14b}{9a} = k$, in two different forms. First, multiply both sides by b and subtract a to obtain

$$\frac{14b^2}{9a} = bk - a.$$

Because a, b, and k are integers, $14b^2$ must be a multiple of a, and because a and b have no common factors greater than 1, it follows that 14 is divisible by a. Next, multiply both sides of the original equation by $9a$ and subtract $14b$ to obtain

$$\frac{9a^2}{b} = 9ak - 14b.$$

This shows that $9a^2$ is a multiple of b, so 9 must be divisible by b. Thus if (a, b) is a solution, then $b = 1$, 3, or 9, and $a = 1$, 2, 7, or 14. This gives a total of twelve possible solutions (a, b), each of which can be checked quickly. The only such pairs for which

$$\frac{a}{b} + \frac{14b}{9a}$$

is an integer are when (a, b) is $(1, 3)$, $(2, 3)$, $(7, 3)$, or $(14, 3)$.

Difficulty: Hard

NCTM Standard: Algebra for Grades 9-12: Analyze change in various contexts .

Mathworld.com Classification: Number Theory > Prime Numbers > Prime Factorization > Factor