

# Correspondence

## Variational Formulation of the Equations for Small Fields Superposed on Finite Biasing Fields in an Electroelastic Body

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**Abstract**—The equations for small fields superposed on finite biasing fields in an electroelastic body are obtained by an expansion procedure in the variational formulation of a nonlinear electroelastic body. The resulting equations, written with respect to the reference configuration, are the same as those reported earlier except for some algebraic errors that have been corrected in the paper. A variational principle for the incremental fields naturally results from the procedure.

### I. INTRODUCTION

THE linear theory of piezoelectricity assumes infinitesimal deviation from a reference state in which there are no pre-existing mechanical and/or electrical fields, commonly referred to as initial or biasing fields. The presence of biasing fields causes a material to behave differently apparently, and renders the linear theory of piezoelectricity invalid. Description of the behavior of electroelastic materials in the presence of finite biasing fields requires the theory for infinitesimal fields superposed on finite biasing fields [1], which is a consequence of the fully nonlinear theory of electroelasticity [2]. Knowledge of the effects of biasing fields in an electroelastic body has important applications in frequency stability analysis of piezoelectric resonators used in time keeping and communication devices [3], piezoelectric acoustic wave sensors [4], buckling of thin piezoelectric structures [5], measurement of nonlinear electroelastic material constants by acoustic waves propagating in an electroelastic body under biasing mechanical and/or electric fields [6], and modeling of electrostrictive materials that operate under a biasing electric field [7]. The equations governing small incremental fields superposed on finite biasing fields in an electroelastic body can be derived from the nonlinear equations of electroelasticity by an expansion procedure based on the smallness of the incremental fields [1]. The derivation is direct but is rather lengthy. Early equations were written with respect to the initially deformed configuration [1], [8]. It was later found that, for many applications, equations written with respect to the reference configuration without initial deformations and fields were more convenient and were given

in [3]. In this paper a different procedure for deriving the equations governing the incremental fields is given. It is based on the variational formulation of nonlinear electroelasticity. The variational principle is written with respect to the reference configuration. As a result, the equations obtained also are written with respect to the reference configuration. In addition, a variational principle for the incremental fields naturally results from the procedure, whose stationary condition yields the governing equations for the incremental fields. A total energy density is used, which makes the derivation procedure very concise.

### II. VARIATIONAL FORMULATION OF NONLINEAR ELECTROELASTICITY

Consider an electroelastic body that, in the reference configuration, occupies a region  $V$  with boundary surface  $S$ . For mechanical boundary conditions,  $S$  is partitioned into  $S_y$  and  $S_T$ , on which motion (or displacement) and traction are prescribed, respectively. Electrically  $S$  is partitioned into  $S_\phi$  and  $S_\sigma$  with prescribed electric potential and surface-free charge, respectively:

$$\begin{aligned} S_y \cup S_T &= S_\phi \cup S_\sigma = S, \\ S_y \cap S_T &= S_\phi \cap S_\sigma = 0. \end{aligned} \quad (1)$$

The deformation of the body is described by  $y_i = y_i(X_L, t)$ , where  $y_i$  denotes the present coordinates and  $X_L$  the reference coordinates of material points with respect to the same Cartesian coordinate system. The Cartesian tensor notation, the summation convention for repeated tensor indices, and the convention that a comma followed by an index denotes partial differentiation with respect to the coordinate associated with the index will be used. A superimposed dot represents material time derivative. The stationary condition of the following functional gives the equations governing the nonlinear behavior of an electroelastic body [9]:

$$\begin{aligned} \Pi(y_i, \phi) &= \int_{t_0}^t dt \int_V \left[ \frac{1}{2} \rho_0 \dot{y}_i \dot{y}_i - \rho_0 \pi(E_{KL}, W_K) \right. \\ &\quad \left. + \rho_0 f_i y_i - \rho_E \phi \right] dV \\ &\quad + \int_{t_0}^t dt \int_{S_T} \bar{T}_i y_i dS - \int_{t_0}^t dt \int_{S_\sigma} \bar{\sigma} \phi dS, \end{aligned} \quad (2)$$

where  $\rho_0$  is the mass density in the reference configuration,  $\rho_E$  is the free-charge density per unit undeformed volume,  $f_k$  is the body force per unit mass,  $\bar{T}_i$  is the surface traction per unit undeformed area,  $\bar{\sigma}$  is the surface free charge per unit undeformed area,  $\phi$  is the electric potential, and  $\pi$  is an energy density per unit mass (the total energy density

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in [9]).  $\pi$  is a function of  $y_i$  and  $\phi$  through the finite strain tensor  $E_{KL}$  and the reference electric potential gradient  $W_K$  given below, and the admissible  $y_i$  and  $\phi$  for  $\Pi$  satisfy the following boundary conditions on  $S_y$  and  $S_\phi$ :

$$\begin{aligned} E_{KL} &= (y_{i,K}y_{i,L} - \delta_{KL})/2, \quad \text{in } V, \\ W_K &= -\phi_{,K} \quad \text{in } V, \\ y_i &= \bar{y}_i \quad \text{on } S_y, \\ \phi &= \bar{\phi} \quad \text{on } S_\phi, \end{aligned} \quad (3)$$

where  $\delta_{KL}$  is the Kronecker delta, and  $\bar{y}_i$  and  $\bar{\phi}$  are prescribed boundary motion and potential. The energy density  $\pi$  that determines the constitutive relations of nonlinear electroelastic materials may be written as [10]:

$$\begin{aligned} &\rho_0\pi(E_{KL}, W_K) + \frac{1}{2}\varepsilon_0 J E_k E_k \\ &= \frac{1}{2}c_{2ABCD} E_{AB}E_{CD} - e_{ABC}W_A E_{BC} - \frac{1}{2}\chi_{2AB} W_A W_B \\ &+ \frac{1}{6}c_{3ABCDEF} E_{AB}E_{CD}E_{EF} + \frac{1}{2}k_{1ABCDE} W_A E_{BC} E_{DE} \\ &- \frac{1}{2}b_{ABCD}W_A W_B E_{CD} - \frac{1}{6}\chi_{3ABC} W_A W_B W_C \\ &+ \frac{1}{24}c_{4ABCDEF GH} E_{AB}E_{CD}E_{EF}E_{GH} \\ &+ \frac{1}{6}k_{2ABCDEF G} W_A E_{BC} E_{DE} E_{FG} \\ &+ \frac{1}{4}a_{1ABCDEF} W_A W_B E_{CD} E_{EF} \\ &+ \frac{1}{6}k_{3ABCDE} W_A W_B W_C E_{DE} \\ &- \frac{1}{24}\chi_{4ABCD} W_A W_B W_C W_D + \dots, \end{aligned} \quad (4)$$

where  $J = \det(y_{k,L})$ ,  $E_k = -\phi_{,k}$  and  $\varepsilon_0$  is the dielectric permittivity of free space. The material constants:

$$\begin{aligned} &c_{2ABCD}, e_{ABC}, \chi_{2AB}, \\ &c_{3ABCDEF}, k_{1ABCDE}, b_{ABCD}, \chi_{3ABC}, \\ &c_{4ABCDEF GH}, k_{2ABCDEF G}, a_{1ABCDEF}, k_{3ABCDE}, \chi_{4ABCD}, \end{aligned} \quad (5)$$

are the second-order elastic, piezoelectric, electric permeability, third-order elastic, first odd electroelastic, electrostrictive, third-order electric permeability, fourth-order elastic, second odd electroelastic, first even electroelastic, third odd electroelastic, and fourth-order electric permeability, respectively. These material constants are called the fundamental material constants. The second-order constants are responsible for linear material behaviors. The third- and higher-order material constants are related to nonlinear behaviors of materials. The stationary condi-

tion of  $\Pi$  implies [9] the following equations and natural boundary conditions of nonlinear electroelasticity:

$$\begin{aligned} K_{Lk,L} + \rho_0 f_k &= \rho_0 \ddot{y}_k, \quad \text{in } V, \\ \Delta_{K,K} &= \rho E, \quad \text{in } V, \\ K_{Lk} N_L &= \bar{T}_k \quad \text{on } S_T, \\ \Delta_K N_K &= -\bar{\sigma} \quad \text{on } S_\sigma, \end{aligned} \quad (6)$$

where  $N_L$  is the unit exterior normal of  $S$ .  $K_{Lk}$  and  $\Delta_K$  are the first Piola-Kirchhoff stress tensor and the reference electric displacement vector, respectively, and are defined from  $\pi$  by the following constitutive relations:

$$\begin{aligned} K_{Lk} &= y_{k,K} \rho_0 \partial \pi / \partial E_{KL}, \\ \Delta_K &= -\rho_0 \partial \pi / W_K. \end{aligned} \quad (7)$$

Eq. (3) and (6) form the boundary-value problem of nonlinear electroelasticity, with additional initial conditions if a dynamic problem is under consideration. They can be written as four equations for  $y_i(\mathbf{X}, t)$  and  $\phi(\mathbf{X}, t)$ . The above procedure of using the total energy density  $\pi$  is compact and concise. Next we partition  $\pi$  into:

$$-\rho_0\pi(E_{KL}, W_K) = -\rho_0\psi(E_{KL}, W_K) + e(E_{KL}, W_K), \quad (8)$$

where we have denoted:

$$\begin{aligned} 2e(E_{KL}, W_K) &= \varepsilon_0 J E_k E_k = \varepsilon_0 J \phi_{,k} \phi_{,k} \\ &= \varepsilon_0 J X_{L,k} X_{M,k} \phi_{,L} \phi_{,M} \\ &= \varepsilon_0 J C_{LM}^{-1} W_L W_M, \\ C_{KL} &= y_{k,K} y_{k,L} = 2E_{KL} + \delta_{KL}, \\ C_{KL}^{-1} C_{LM} &= \delta_{KM}, \quad J_C = \det(C_{KL}) = J^2. \end{aligned} \quad (9)$$

Physically,  $e$  represents pure electrostatic energy per unit undeformed volume.  $\psi$  is a free-energy density per unit mass. Comparing (8) with (4), we find:

$$\begin{aligned} \rho_0\psi(E_{KL}, W_K) &= \frac{1}{2}c_{2ABCD} E_{AB}E_{CD} - e_{ABC}W_A E_{BC} \\ &- \frac{1}{2}\chi_{2AB} W_A W_B + \frac{1}{6}c_{3ABCDEF} E_{AB}E_{CD}E_{EF} \\ &+ \frac{1}{2}k_{1ABCDE} W_A E_{BC} E_{DE} - \frac{1}{2}b_{ABCD}W_A W_B E_{CD} \\ &- \frac{1}{6}\chi_{3ABC} W_A W_B W_C \\ &+ \frac{1}{24}c_{4ABCDEF GH} E_{AB}E_{CD}E_{EF}E_{GH} \\ &+ \frac{1}{6}k_{2ABCDEF G} W_A E_{BC} E_{DE} E_{FG} \\ &+ \frac{1}{4}a_{1ABCDEF} W_A W_B E_{CD} E_{EF} \\ &+ \frac{1}{6}k_{3ABCDE} W_A W_B W_C E_{DE} \\ &- \frac{1}{24}\chi_{4ABCD} W_A W_B W_C W_D + \dots. \end{aligned} \quad (10)$$

With the following partial derivative of  $e$  [9]:

$$\begin{aligned}\partial e / \partial E_{KL} &= -J X_{K,k} X_{L,l} \varepsilon_0 (E_k E_l - E_m E_m \delta_{kl} / 2), \\ \partial e / \partial W_K &= J X_{K,k} \varepsilon_0 E_k.\end{aligned}\quad (11)$$

The constitutive relations in (7) take the following form when  $\psi$  is used:

$$\begin{aligned}K_{Lk} &= y_{k,K} \rho_0 \partial \psi / \partial E_{KL} + J X_{L,j} \varepsilon_0 (E_j E_k - E_i E_i \delta_{jk} / 2), \\ \Delta_K &= -\rho_0 \partial \psi / \partial W_K + J X_{K,k} \varepsilon_0 E_k.\end{aligned}\quad (12)$$

Eq. (12) is exactly the same as what is given in [3]. The partition of energy in (8) implies the separation of the Maxwell's stress tensor [the last term in (12)<sub>1</sub>] and the reference electric field vector [the last term in (12)<sub>2</sub>], which is physically more revealing.

### III. BIASING AND INCREMENTAL FIELDS

Consider an electroelastic body. We distinguish the following three configurations.

#### A. The Reference Configuration

In the reference state, the body is undeformed and free of all fields. A generic point at this state is denoted by  $\mathbf{X}$  with rectangular coordinates  $X_K$ . The mass density in the reference configuration is denoted by  $\rho_0$ .

#### B. The Initial Configuration

In this state the body is under the action of body force  $f_\alpha^{(0)}$ , body free charge  $\rho_E^{(0)}$ , prescribed surface motion  $\bar{\xi}_\alpha$ , traction  $\bar{T}_\alpha^{(0)}$ , potential  $\bar{\phi}^{(0)}$ , and free charge  $\bar{\sigma}^{(0)}$ . The body has deformed finitely and carries finite electric fields. The position of the material point associated with  $\mathbf{X}$  is given by  $\xi_\alpha = \xi_\alpha(\mathbf{X}, t)$ , with the Jacobian of the initial deformation denoted by  $J^{(0)} = \det(\xi_{\alpha,L})$ . The electric potential of this state is denoted by  $\phi^{(0)}(\mathbf{X}, t)$ , and the electric field by  $E_\alpha^{(0)} = -\phi_{,\alpha}^{(0)}$ . The initial deformations and fields  $\xi_\alpha(\mathbf{X}, t)$  and  $\phi^{(0)}(\mathbf{X}, t)$  satisfy the equations of nonlinear electroelasticity (3) and (6) under  $f_\alpha^{(0)}$ ,  $\rho_E^{(0)}$ ,  $\bar{\xi}_\alpha$ ,  $\bar{T}_\alpha^{(0)}$ ,  $\bar{\phi}^{(0)}$ , and  $\bar{\sigma}^{(0)}$ . The solution to the initial deformation problem is assumed known.

#### C. The Present Configuration

To the deformed body at the initial configuration, infinitesimal deformations and electric fields are applied. The body is under the action of  $f_i$ ,  $\rho_E$ ,  $\bar{y}_i$ ,  $\bar{T}_i$ ,  $\bar{\phi}$ , and  $\bar{\sigma}$ . The final position of the material point associated with  $\mathbf{X}$  is given by  $y_i(\mathbf{X}, t)$ , with electric potential  $\phi(\mathbf{X}, t)$ .  $y_i(\mathbf{X}, t)$  and  $\phi(\mathbf{X}, t)$  also satisfy (3) and (6).

### IV. VARIATIONAL PRINCIPLE AND GOVERNING EQUATIONS FOR THE INCREMENTAL FIELDS

The incremental deformation is assumed to be infinitesimal; therefore we write:

$$y_i = \delta_{i\alpha}(\xi_\alpha + \varepsilon u_\alpha), \quad (13)$$

where  $\varepsilon$  is a small and dimensionless number,  $\delta_{i\alpha}$  is the Kronecker delta, and  $\varepsilon \mathbf{u}$  is the small incremental displacement vector. Corresponding to (13), other quantities of the present state can be written as:

$$\begin{aligned}\phi &= \phi^{(0)} + \varepsilon \phi^{(1)} + \varepsilon^2 \phi^{(2)} \dots, \\ f_i &= \delta_{i\alpha} (f_\alpha^{(0)} + \varepsilon f_\alpha^{(1)} + \varepsilon^2 f_\alpha^{(2)} \dots), \\ \rho_E &= \rho_E^{(0)} + \varepsilon \rho_E^{(1)} + \varepsilon^2 \rho_E^{(2)} \dots, \\ \bar{y}_i &= \delta_{i\alpha} (\bar{\xi}_\alpha + \varepsilon \bar{u}_\alpha), \\ \bar{T}_i &= \delta_{i\alpha} (\bar{T}_\alpha^{(0)} + \varepsilon \bar{T}_\alpha^{(1)} + \varepsilon^2 \bar{T}_\alpha^{(2)} \dots), \\ \bar{\phi} &= \bar{\phi}^{(0)} + \varepsilon \bar{\phi}^{(1)} + \varepsilon^2 \bar{\phi}^{(2)} \dots, \\ \bar{\sigma} &= \bar{\sigma}^{(0)} + \varepsilon \bar{\sigma}^{(1)} + \varepsilon^2 \bar{\sigma}^{(2)} \dots,\end{aligned}\quad (14)$$

where, due to nonlinearity, higher powers of  $\varepsilon$  may arise. We want to derive equations governing  $\mathbf{u}$  and  $\phi^{(1)}$ . From (13) and (14), we can further write:

$$\begin{aligned}E_{KL} &= E_{KL}^{(0)} + \varepsilon E_{KL}^{(1)} + \varepsilon^2 E_{KL}^{(2)}, \\ W_K &= W_K^{(0)} + \varepsilon W_K^{(1)} + \varepsilon^2 W_K^{(2)} \dots,\end{aligned}\quad (15)$$

where

$$\begin{aligned}E_{KL}^{(0)} &= (\xi_{\alpha,K} \xi_{\alpha,L} - \delta_{KL}) / 2, \\ E_{KL}^{(1)} &= (\xi_{\alpha,K} u_{\alpha,L} + \xi_{\alpha,L} u_{\alpha,K}) / 2, \\ E_{KL}^{(2)} &= u_{\alpha,K} u_{\alpha,L} / 2, \\ W_K^{(0)} &= -\phi_{,K}^{(0)}, \\ W_K^{(1)} &= -\phi_{,K}^{(1)}, \\ W_K^{(2)} &= -\phi_{,K}^{(2)}.\end{aligned}\quad (16)$$

Substituting (13)–(16) into  $\Pi$  in (2), we obtain:

$$\Pi = \Pi^{(0)} + \varepsilon \Pi^{(1)} + \varepsilon^2 \Pi^{(2)} \dots, \quad (17)$$

where

$$\begin{aligned}\Pi^{(0)} &= \int_{t_0}^t dt \int_V \left[ \rho_0 \dot{\xi}_\alpha \dot{\xi}_\alpha / 2 - \rho_0 \pi (E_{KL}^{(0)}, W_K^{(0)}) \right. \\ &\quad \left. + \rho_0 f_\alpha^{(0)} \xi_\alpha - \rho_E^{(0)} \phi^{(0)} \right] dV \\ &\quad + \int_{t_0}^t dt \int_{S_T} \bar{T}_\alpha^{(0)} \xi_\alpha dS - \int_{t_0}^t dt \int_{S_\sigma} \bar{\sigma}^{(0)} \phi^{(0)} dS,\end{aligned}\quad (18)$$

$$\begin{aligned} \Pi^{(1)} = & \int_{t_0}^t dt \int_V \left[ \rho_0 \dot{\xi}_\alpha \dot{u}_\alpha - \rho_0 \frac{\partial \pi}{\partial E_{KL}} \Big|_{E_{KL}^{(0)}, W_K^{(0)}} E_{KL}^{(1)} \right. \\ & - \rho_0 \frac{\partial \pi}{\partial W_K} \Big|_{E_{KL}^{(0)}, W_K^{(0)}} W_K^{(1)} \\ & + \rho_0 f_\alpha^{(0)} u_\alpha + \rho_0 f_\alpha^{(1)} \xi_\alpha - \rho_E^{(0)} \phi^{(1)} - \rho_E^{(1)} \phi^{(0)} \Big] dV \quad (19) \\ & + \int_{t_0}^t dt \int_{S_T} (\bar{T}_\alpha^{(0)} u_\alpha + \bar{T}_\alpha^{(1)} \xi_\alpha) dS \\ & - \int_{t_0}^t dt \int_{S_\sigma} (\bar{\sigma}^{(0)} \phi^{(1)} + \bar{\sigma}^{(1)} \phi^{(0)}) dS, \end{aligned}$$

and

$$\begin{aligned} \Pi^{(2)} = & \int_{t_0}^t dt \int_V \left[ \frac{1}{2} \rho_0 \dot{u}_\alpha \dot{u}_\alpha - \rho_0 \frac{\partial \pi}{\partial E_{KL}} \Big|_{E_{KL}^{(0)}, W_K^{(0)}} E_{KL}^{(2)} \right. \\ & - \rho_0 \frac{\partial \pi}{\partial W_K} \Big|_{E_{KL}^{(0)}, W_K^{(0)}} W_K^{(2)} \\ & - \frac{1}{2} \rho_0 \frac{\partial^2 \pi}{\partial E_{KL} \partial E_{MN}} \Big|_{E_{KL}^{(0)}, W_K^{(0)}} E_{KL}^{(1)} E_{MN}^{(1)} \\ & - \rho_0 \frac{\partial^2 \pi}{\partial W_M \partial E_{KL}} \Big|_{E_{KL}^{(0)}, W_K^{(0)}} W_M^{(1)} E_{KL}^{(1)} \\ & - \frac{1}{2} \rho_0 \frac{\partial^2 \pi}{\partial W_M \partial W_N} \Big|_{E_{KL}^{(0)}, W_K^{(0)}} W_M^{(1)} W_N^{(1)} + \rho_0 f_\alpha^{(1)} u_\alpha^{(20)} \\ & + \rho_0 f_\alpha^{(2)} \xi_\alpha - \rho_E^{(0)} \phi^{(2)} - \rho_E^{(1)} \phi^{(1)} - \rho_E^{(2)} \phi^{(0)} \Big] dV \\ & + \int_{t_0}^t dt \int_{S_T} (\bar{T}_\alpha^{(1)} u_\alpha + \bar{T}_\alpha^{(2)} \xi_\alpha) dS \\ & - \int_{t_0}^t dt \int_{S_\sigma} (\bar{\sigma}^{(0)} \phi^{(2)} + \bar{\sigma}^{(1)} \phi^{(1)} + \bar{\sigma}^{(2)} \phi^{(0)}) dS. \end{aligned}$$

Comparing (18) to (2), we recognize (18) to be simply the variational functional for the initial deformation.

Using the fact that the initial deformation satisfies (3) and (6),  $\Pi^{(1)}$  in (19) can be written into the following much simpler form:

$$\begin{aligned} \Pi^{(1)} = & \int_{t_0}^t dt \int_V (\rho_0 f_\alpha^{(1)} \xi_\alpha - \rho_E^{(1)} \phi^{(0)}) dV \\ & + \int_{t_0}^t dt \int_{S_T} \bar{T}_\alpha^{(1)} \xi_\alpha dS - \int_{t_0}^t dt \int_{S_\sigma} \bar{\sigma}^{(1)} \phi^{(0)} dS, \quad (21) \end{aligned}$$

which does not depend on  $\mathbf{u}$  and  $\phi^{(1)}$  anymore. If  $f_\alpha^{(0)}, \rho_E^{(0)}, \bar{T}_\alpha^{(0)}$ , and  $\bar{\sigma}^{(0)}$  were held constant—or, in other words— $f_\alpha^{(1)} = \rho_E^{(1)} = \bar{T}_\alpha^{(1)} = \bar{\sigma}^{(1)} = 0$ , then  $\Pi^{(1)} = 0$ , which simply shows that  $\Pi^{(0)}$  is the variational functional for the initial deformation.

We are interested in equations for the first order incremental fields  $\mathbf{u}$  and  $\phi^{(1)}$ . Therefore, we drop all second

order quantities  $\phi^{(2)}, f_\alpha^{(2)}, \rho_E^{(2)}, \bar{T}_\alpha^{(2)}$ , and  $\bar{\sigma}^{(2)}$  in  $\Pi^{(2)}$  and obtain:

$$\begin{aligned} \Gamma(\mathbf{u}, \phi^{(1)}) = & \int_{t_0}^t dt \int_V \left[ \rho_0 \dot{u}_\alpha \dot{u}_\alpha / 2 - G_{K\alpha L\gamma} u_{K,\alpha} u_{L,\gamma} / 2 \right. \\ & - R_{KL\gamma} \phi_{,K}^{(1)} u_{L,\gamma} + L_{KL} \phi_{,K}^{(1)} \phi_{,L}^{(1)} / 2 \\ & + \rho_0 f_\alpha^{(1)} u_\alpha - \rho_E^{(1)} \phi^{(1)} \Big] dV \quad (22) \\ & + \int_{t_0}^t dt \int_{S_T} \bar{T}_\alpha^{(1)} u_\alpha dS - \int_{t_0}^t dt \int_{S_\sigma} \bar{\sigma}^{(1)} \phi^{(1)} dS, \end{aligned}$$

where

$$\begin{aligned} G_{K\alpha L\gamma} = & \xi_{\alpha,M} \rho_0 \frac{\partial^2 \pi}{\partial E_{KM} \partial E_{LN}} \Big|_{E_{KL}^{(0)}, W_K^{(0)}} \xi_{\gamma,N} \\ & + \rho_0 \frac{\partial \pi}{\partial E_{KL}} \Big|_{E_{KL}^{(0)}, W_K^{(0)}} \delta_{\alpha\gamma} = G_{L\gamma K\alpha}, \\ R_{KL\gamma} = & -\rho_0 \frac{\partial^2 \pi}{\partial W_K \partial E_{ML}} \Big|_{E_{KL}^{(0)}, W_K^{(0)}} \xi_{\gamma,M}, \quad (23) \\ L_{KL} = & -\rho_0 \frac{\partial^2 \pi}{\partial W_K \partial W_K} \Big|_{E_{KL}^{(0)}, W_K^{(0)}} = L_{LK}, \end{aligned}$$

are the effective elastic, piezoelectric, and dielectric constants. They are functions of linear and nonlinear material constants through  $\pi$ . They also depend on the biasing fields (assumed known). We note that the integrand of the volume integral in (22) is quadratic in the gradients of  $\mathbf{u}$  and  $\phi^{(1)}$ . The admissible  $\mathbf{u}$  and  $\phi^{(1)}$  for  $\Gamma$  in (22) must satisfy:

$$u_\alpha = \bar{u}_\alpha \text{ on } S_y, \quad \phi^{(1)} = \bar{\phi}^{(1)} \text{ on } S_\phi. \quad (24)$$

The first variation of  $\Gamma$  is found to be:

$$\begin{aligned} \delta\Gamma(\mathbf{u}, \phi^{(1)}) = & \int_{t_0}^t dt \int_V \left[ (K_{L\alpha,L}^{(1)} + \rho_0 f_\alpha^{(1)} - \rho_0 \ddot{u}_\alpha) \delta u_\alpha \right. \\ & + (\Delta_{K,K}^{(1)} - \rho_E^{(1)}) \delta \phi^{(1)} \Big] dV \\ & + \int_{t_0}^t dt \int_{S_T} (\bar{T}_\alpha^{(1)} - K_{L\alpha}^{(1)} N_L) \delta u_\alpha dS \quad (25) \\ & - \int_{t_0}^t dt \int_{S_\sigma} (\Delta_K^{(1)} N_K + \bar{\sigma}^{(1)}) \delta \phi^{(1)} dS, \end{aligned}$$

where we have introduced and denoted the incremental stress tensor and electric displacement vector by the following effective constitutive relations for the incremental fields:

$$\begin{aligned} K_{K\alpha}^{(1)} = & G_{K\alpha L\gamma} u_{\gamma,L} + R_{LK\alpha} \phi_{,L}^{(1)}, \quad (26) \\ \Delta_K^{(1)} = & R_{KL\gamma} u_{\gamma,L} - L_{KL} \phi_{,L}^{(1)}. \end{aligned}$$

We note that in (26) the incremental stress tensor and electric displacement vector depend linearly on the incremental displacement gradient and potential gradient. The

stationary condition of  $\Gamma$  then implies the following equations and natural boundary conditions for the incremental fields:

$$\begin{aligned} K_{K\alpha,K}^{(1)} + \rho_0 f_\alpha^{(1)} &= \rho_0 \ddot{u}_\alpha, \quad \text{in } V, \\ \Delta_{K,K}^{(1)} &= \rho_E^{(1)}, \quad \text{in } V, \\ K_{L\alpha}^{(1)} N_L &= \bar{T}_\alpha^{(1)} \quad \text{on } S_T, \\ \Delta_K^{(1)} N_K &= -\bar{\sigma}^{(1)} \quad \text{on } S_\sigma, \end{aligned} \quad (27)$$

which, together with (24), form the boundary-value problem for the incremental fields. If the partition of energy (8) is introduced into (23), we have:

$$\begin{aligned} G_{K\alpha L\gamma} &= \xi_{\alpha,M} \rho_0 \frac{\partial^2 \psi}{\partial E_{KM} \partial E_{LN}} \Big|_{E_{KL}^{(0)}, W_K^{(0)}} \xi_{\gamma,N} \\ &+ \rho_0 \frac{\partial \psi}{\partial E_{KL}} \Big|_{E_{KL}^{(0)}, W_K^{(0)}} \delta_{\alpha\gamma} + g_{K\alpha L\gamma}, \\ R_{KL\gamma} &= -\rho_0 \frac{\partial^2 \psi}{\partial W_K \partial E_{ML}} \Big|_{E_{KL}^{(0)}, W_K^{(0)}} \xi_{\gamma,M} + r_{KL\gamma}, \quad (28) \\ L_{KL} &= -\rho_0 \frac{\partial^2 \psi}{\partial W_K \partial W_L} \Big|_{E_{KL}^{(0)}, W_K^{(0)}} + l_{KL}, \end{aligned}$$

where

$$\begin{aligned} g_{K\alpha L\gamma} &= \varepsilon_0 J^{(0)} \left[ E_\alpha^{(0)} E_\beta^{(0)} (X_{K,\beta} X_{L,\gamma} - X_{K,\gamma} X_{L,\beta}) \right. \\ &+ E_\beta^{(0)} E_\gamma^{(0)} (X_{K,\alpha} X_{L,\beta} - X_{K,\beta} X_{L,\alpha}) \\ &+ E_\beta^{(0)} E_\beta^{(0)} (X_{K,\gamma} X_{L,\alpha} - X_{K,\alpha} X_{L,\gamma})/2 \\ &\left. - E_\alpha^{(0)} E_\gamma^{(0)} X_{K,\beta} X_{L,\beta} \right], \quad (29) \\ r_{KL\gamma} &= \varepsilon_0 J^{(0)} (E_\alpha^{(0)} X_{K,\alpha} X_{L,\gamma} \\ &- E_\alpha^{(0)} X_{K,\gamma} X_{L,\alpha} - E_\gamma^{(0)} X_{K,\alpha} X_{L,\alpha}), \\ l_{KL} &= \varepsilon_0 J^{(0)} X_{K,\alpha} X_{L,\alpha}, \end{aligned}$$

which are the equations given in [3]. We note that the equation in [3] corresponding to (29)<sub>1</sub> has been found to be with some algebraic errors [11], while (29)<sub>1</sub> is believed to be correct.

## V. CONCLUSIONS

From the variational formulation of nonlinear electroelasticity, by a systematic expansion procedure, a variational principle is obtained for infinitesimal incremental deformations and fields superposed on finite biasing deformations and fields in an electroelastic body. The stationary condition of the derived variational principle yields the governing equations for the incremental deformations and fields. The total energy formulation simplifies the derivation procedure.

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