Letters

On the Effect of the Electric Field in the Free Space Surrounding a Finite Piezoelectric Body

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***Abstract*—This note shows that a widely used approximation in analyzing motions of a finite piezoelectric body (i.e., the normal component of the electric displacement vector at an unelectroded material-air interface can be taken to be zero when the dielectric constant of the material is much larger than the electric permittivity of the free space) may lead to inaccurate results.**

I. Introduction

In the study of motions of a finite piezoelectric body, at an unelectroded material-air interface, the continuity of normal **D** (the electric displacement) and the continuity of tangential **E** (the electric field) between the fields in the material body and the fields in the surrounding free space are required in an exact description [1]. It has been widely believed that, when the appropriate dielectric constant of the material is much larger than the electric permittivity of the free space, the normal component $D_i n_i$ of the electric displacement vector at an interface with a normal **n** can be approximately taken to be zero [2]. Mathematically, with this approximation, the problem of a finite material body imbedded in the free space with interface continuity conditions reduces to a boundary value problem over the finite domain occupied by the material body, and thus the problem is greatly simplified. The approximation has been widely used. The references are too numerous to list. In this approximation, the continuity of tangential **E** at the interface is dropped, and mathematically the continuity of tangential **E** is not needed to form a well-posed boundary value problem over the material region.

We wonder about the effect of dropping the continuity of tangential **E** under the approximation of $D_i n_i = 0$, and we examine two simple problems in which a strong presence of a tangential **E** at an unelectroded material-air interface exists. Results show that the approximation of $D_i n_i = 0$ does not always yield results that are close to the results obtained by properly considering the electric field in the free space with interface continuity conditions.

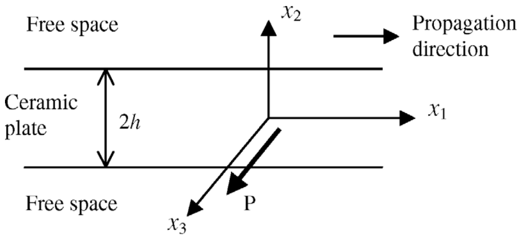

Fig. 1. A piezoelectric plate of polarized ceramics.

II. Antiplane Waves in a Plate

Consider the propagation of antiplane or shear horizontal (SH) waves in an unbounded plate of polarized ceramics [3] (see Fig. 1). The boundary-value problem is [3]:

$$\begin{gathered}
T_{ji,j} = \rho \ddot{u}_i, \quad D_{i,i} = 0, \quad -h < x_2 < h, \\
T_{ij} = c_{ijkl} S_{kl} - e_{kij} E_k, \quad D_i = e_{ikl} S_{kl} + \varepsilon_{ik} E_k, \\
-h < x_2 < h, \\
S_{ij} = \left(u_{i,j} + u_{j,i} \right)/2, \quad E_i = -\phi_{,i}, \\
-h < x_2 < h, \\
D_{i,i} = 0, \quad D_i = \varepsilon_0 E_i, \quad E_i = -\phi_{,i}, \quad |x_2| > h, \\
T_{2j}\left(x_3 = \pm h^- \right) = 0, \\
\phi\left(x_2 = \pm h^- \right) = \phi\left(x_2 = \pm h^+ \right), \\
D_2\left(x_2 = \pm h^- \right) = D_2\left(x_2 = \pm h^+ \right),
\end{gathered} \tag{1}$$

where u_i is the mechanical displacement vector, T_{ij} is the stress tensor, S_{ij} is the strain tensor, E_i is the electric field, D_i is the electric displacement, ϕ is the electric potential. The coefficients c_{ijkl}, e_{kij}, and ε_{ij} are the elastic, piezoelectric, and dielectric constants. ρ is the mass density, and ε_0 is the electric permittivity of free space. The summation convention for repeated tensor indices and the convention that a comma followed by an index denotes partial differentiation with respect to the coordinate associated with the index are used. A superimposed dot represents a time derivative. Note that the electric field in the free space is properly considered in (1), and there are continuity conditions at the material-air interfaces in (1). In particular, the continuity of the electric potential at the interfaces at $x_2 = \pm h$ implies the continuity of tangential **E** there.

We are interested in antiplane motion with:

$$\begin{aligned}
&u_1 = u_2 = 0, \quad u_3 = u_3(x_2)\cos(\xi x_1 - \omega t), \\
&\phi = \phi(x_2)\cos(\xi x_1 - \omega t),
\end{aligned} \tag{2}$$

where ξ is the wave number and ω is the frequency. Note that the ϕ in (2) implies the presence of E_1 and E_2, and E_1 is tangential at the material-air interfaces. Solutions to (1) in the form of (2) can be classified as symmetric and antisymmetric waves. The dispersion relation, a relation

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between ξ and ω, of the symmetric waves is determined by [3]:

$$\left(\Omega^2-Z^2\right)^{1/2}\left[1+\frac{\varepsilon_{11}}{\varepsilon_0}\tanh\left(\frac{\pi}{2}Z\right)\right]\tan\left[\frac{\pi}{2}\left(\Omega^2-Z^2\right)^{1/2}\right] = -\bar{k}_{15}^2 Z\tanh\left(\frac{\pi}{2}Z\right), \quad (3)$$

where the dimensionless frequency Ω and the dimensionless wave number Z are defined by:

$$\Omega^2=\omega^2\Big/\left(\frac{\pi^2\bar{c}_{44}}{4\rho h^2}\right),\quad Z=\xi\Big/\left(\frac{\pi}{2h}\right), \quad (4)$$

and:

$$\bar{c}_{44}=c_{44}+e_{15}^2/\varepsilon_{11},\quad \bar{k}_{15}^2=e_{15}^2/\left(\bar{c}_{44}\varepsilon_{11}\right). \quad (5)$$

If we adopt the approximation of $D_2=0$ at the plate surfaces, the boundary value problem is:

$$\begin{gathered}T_{ji,j}=\rho\ddot{u}_i,\quad D_{i,i}=0,\quad -h<x_2<h,\\ T_{ij}=c_{ijkl}S_{kl}-e_{kij}E_k,\quad D_i=e_{ikl}S_{kl}+\varepsilon_{ik}E_k,\\ -h<x_2<h,\\ S_{ij}=\left(u_{i,j}+u_{j,i}\right)/2,\quad E_i=-\phi_{,i},\\ -h<x_2<h,\\ T_{2j}\left(x_3=\pm h^-\right)=0,\\ D_2\left(x_3=\pm h^-\right)=0.\end{gathered} \quad (6)$$

Note that in (6) there is no mentioning of the electric field in the free space. Solving (6), one obtains the symmetric wave solution in the form of (2) with the following dispersion relation [3]:

$$\sinh\left(\frac{\pi}{2}Z\right)\sin\left[\frac{\pi}{2}\left(\Omega^2-Z^2\right)^{1/2}\right]=0. \quad (7)$$

Note that (7) also can be obtained from (3) by letting $\varepsilon_{11}\to\infty$ [3]. However, our careful examination of this limit procedure shows that the procedure is questionable when Z is very small, i.e., when long waves are under consideration. In this case $\varepsilon_{11}Z/\varepsilon_0$ may not be much larger than 1, which is needed in the reduction from (3) to (7). For small Z, Ω may or may not be small depending on which branch of the dispersion curves is under consideration. We examine the face-shear wave, which is the lowest order symmetric wave determined by (3) or (7), and for which Ω is also small when Z is small [3]. Then from (3), for small Z and small Ω, we have:

$$\Omega^2\cong Z^2-\bar{k}_{15}^2\frac{Z^2}{1+\dfrac{\varepsilon_{11}}{\varepsilon_0}\dfrac{\pi}{2}Z}, \quad (8)$$

which shows a dispersive wave. From (7) we obtain a nondispersive, face-shear wave with:

$$\Omega=Z. \quad (9)$$

Therefore there is a qualitative difference between the exact solution (3) and the approximate solution (9).

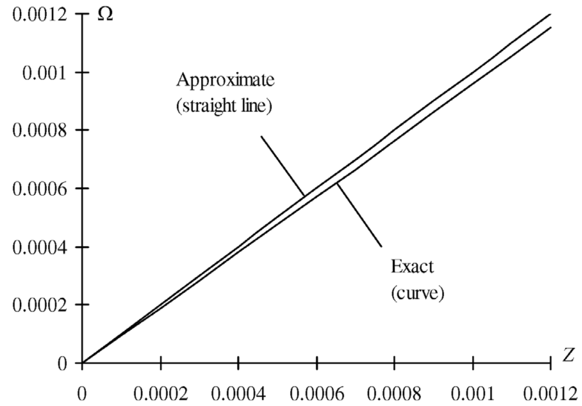

Fig. 2. Dispersion relations of face-shear waves.

To find further details, the dispersion relation of face-shear waves from (3) is determined numerically and is plotted with (9) in Fig. 2. PZT-6B is used in the calculation, which has a fairly large dielectric constant of $\varepsilon_u=407\,\varepsilon_0$. The quantitative difference between the two dispersion curves may or may not be considered as small in different applications. However, it should be emphasized that, in Fig. 2, the straight line is not a tangent of the curve at the origin. Therefore, in this case prescribing $D_2=0$ at the plate surfaces does not yield an approximate dispersion relation for long waves in the sense that, for the long wave limit, the two dispersion relations do not share a common tangent.

III. Antiplane Surface Waves

For antiplane surface waves over a ceramic half space [4], the situation is more interesting. For the case in which the surface of the half space is unelectroded, a surface wave in the form of (2) can be determined with the following speed [4] if the electric field in the free space and the proper material-air interface continuity conditions are considered:

$$v^2=\frac{\omega^2}{\xi^2}=\frac{\bar{c}_{44}}{\rho}\left[1-\frac{\bar{k}_{15}^4}{\left(1+\varepsilon_{11}/\varepsilon_0\right)^2}\right]. \quad (10)$$

However, if $D_i n_i=0$ is prescribed as a boundary condition on the surface of the half space, no surface wave solution can be obtained. Therefore, in this case the exact approach with the proper consideration of the free-space electric field and the approximate approach using $D_i n_i=0$ result in drastically different results. Also note that the limit of (9) when $\varepsilon_{11}\to\infty$ yields the speed of plane shear waves in the ceramic.

IV. Conclusions

In the analysis of motions of a finite piezoelectric body, prescribing $D_i n_i=0$ on an unelectroded material-air interface does not always yield results qualitatively and/or quantitatively close to the solution obtained by properly considering the electric field in the free space and interface continuity conditions.

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