Electrostatic potential energy leading to a gravitational mass change for a system of two point charges

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(Received 1 July 1977; accepted 16 October 1977)

A system consisting of two point charges has an energy contribution from the system electrostatic potential energy and accordingly a contribution to the system mass given by this energy divided by c^2. Here we investigate the change in weight associated with the electrostatic potential energy for a system of two point charges supported side by side against a weak gravitational field. The gravitational distortion of the Coulomb field of a point charge is calculated using the equivalence principle, and it is then shown that the change in the two-particle system weight can be understood precisely as the change in supporting force necessary to balance the electric force of each charge upon the other. The example provides a clear illustration and detailed mechanism for understanding the mass-energy connection for weak gravitational fields.

I. INTRODUCTION

The mass of a system is a measure of its energy content. This fundamental concept for the inertial mass allows a simple illustration[1] in a system of two point charges which are moving side by side and are accelerated in the direction of motion. The electromagnetic forces of each charge on the other provide the detailed mechanism for the increase of inertial mass associated with the system electrostatic energy. However, mass has not only an inertial but also a gravitational aspect. In the present paper we wish to use the same system of two point charges, now located in a weak gravitational field, to give an elementary example linking the system electrostatic potential energy to an increase in the weight of the system. We will use the principle of equivalence to obtain the electromagnetic fields of each point charge supported in the gravitational field, and then will show that the electromagnetic force of each charge on the other accounts in detail for the increase of weight of the system.

II. BASIC ANALYSIS OF THE SYSTEM

The system under consideration consists of two point charges q_1 and q_2 with small masses m_1 and m_2 supported against a weak downwards gravitational field by light strings of equal length attached to a ceiling. Our discussion will be carried out for an observer at rest relative to the charges although one may also consider an observer moving with constant velocity parallel to the gravitational field direction. The weight of the system is given by the vector sum of the supporting external forces on the two charges; the vertical components supporting the charges against gravity will add, while the horizontal components stabilizing the system are equal in magnitude and opposite in direction and hence sum to zero. The (passive) gravitational mass of the system is equal to the system weight divided by the gravitational field strength g.

If only one charge q_1 were present and the other charge were absent, then the force $\mathbf{F}^{(g)}_{\text{ext}\,1}$ necessary to support the charge against the weak gravitational field is

$$\mathbf{F}^{(g)}_{\text{ext}\,1} = m_1 g\hat{i}, \tag{1}$$

where m_1 is the (renormalized) mass of the first particle and $\hat{i}$ is the unit vector upwards in the direction opposite to the gravitational field. Again if charge q_2 were present in the absence of q_1, the supporting force would be

$$\mathbf{F}^{(g)}_{\text{ext}\,2} = m_2 g\hat{i}. \tag{2}$$

The sum of the forces $\mathbf{F}^{(g)}_{\text{ext}\,1}$ and $\mathbf{F}^{(g)}_{\text{ext}\,2}$ gives

$$\mathbf{F}^{(g)}_{\text{ext}\,1} + \mathbf{F}^{(g)}_{\text{ext}\,2} = (m_1 + m_2)\, g\hat{i}. \tag{3}$$

When both particles are present, the rest energy of the system includes the rest energy of each particle m_1c^2, m_2c^2, and also the electrostatic potential energy q_1q_2/l of the system where l is the separation between the particles. Hence it follows that the rest mass M of the system is not simply $(m_1c^2 + m_2c^2)/c^2 = m_1 + m_2$, but rather includes the contribution from the electrostatic potential energy,

$$M = m_1 + m_2 + q_1q_2/lc^2. \tag{4}$$

The external force $\mathbf{F}^{(g)}_{\text{ext tot}}$ needed to support the system weight Mg is accordingly

$$\mathbf{F}^{(g)}_{\text{ext tot}} = (m_1 + m_2 + q_1q_2/lc^2)g\hat{i}. \tag{5}$$

Comparison of Eqs. (3) and (5) shows that an additional external force $\Delta\mathbf{F}^{(g)}_{\text{ext tot}}$ is required to support the mass associated with the electrostatic potential energy of the system,

$$\Delta\mathbf{F}^{(g)}_{\text{ext tot}} = (q_1q_2/lc^2)g\hat{i}. \tag{6}$$

Now the system under our consideration is electromagnetic in nature and so simple that we can analyze in detail all the forces which appear. Thus the forces on charge q_1 in the presence of q_2 include the gravitational force $m_1g(-\hat{i})$, the electromagnetic force $q_1\mathbf{E}_2^{(g)}$ of q_2 on q_1, and the external force $\mathbf{F}^{(g)}_{\text{ext}\,1} + \Delta\mathbf{F}^{(g)}_{\text{ext}\,1}$ of the string on the charge, which external force changes by $\Delta\mathbf{F}^{(g)}_{\text{ext}\,1}$ from the value when q_2 was not present. We are assuming here that the masses m_1 and m_2 are small so that the mutual gravitational attraction may be neglected compared to the electromagnetic forces. The sum of the forces on charge q_1 must lead to equilibrium,

$$\mathbf{F}^{(g)}_{\text{ext}\,1} + \Delta\mathbf{F}^{(g)}_{\text{ext}\,1} + q_1\mathbf{E}_2^{(g)} + m_1g(-\hat{i}) = 0. \tag{7}$$

Comparing this equation with Eq. (1) shows that the change in the external supporting force on q_1 is associated entirely with the electromagnetic force $q_1\mathbf{E}_2^{(g)}$ of q_2 on q_1,

$$\Delta \mathbf{F}^{(g)}_{\text{ext}\,1} = -q_1 \mathbf{E}^{(g)}_2. \tag{8}$$

In exactly analogous fashion we can discuss the equilibrium of charge q_2, and so conclude that an additional external force $\Delta \mathbf{F}^{(g)}_{\text{ext}\,2}$ is necessary to balance the electromagnetic force of q_1 on q_2,

$$\Delta \mathbf{F}^{(g)}_{\text{ext}\,2} = -q_2 \mathbf{E}^{(g)}_1. \tag{9}$$

The change in the total external supporting force is just the sum of the changes for the individual particles

$$\begin{aligned}\Delta \mathbf{F}^{(g)}_{\text{ext tot}} &= \Delta \mathbf{F}^{(g)}_{\text{ext}\,1} + \Delta \mathbf{F}^{(g)}_{\text{ext}\,2} \\ &= -q_1 \mathbf{E}^{(g)}_2 - q_2 \mathbf{E}^{(g)}_1. \end{aligned} \tag{10}$$

However, we have already remarked in Eq. (5) that the change in the external supporting force, which is interpreted as a change in system weight, is associated with the electrostatic potential energy of the system. Our analysis thus shows from Eqs. (6) and (10) that, if all is consistent, the increase of weight of our system is to be understood as due to the electromagnetic force of each charge on the other

$$(q_1 q_2 / l c^2) g \hat{i} = -q_1 \mathbf{E}^{(g)}_2 - q_2 \mathbf{E}^{(g)}_1. \tag{11}$$

III. ELECTRIC FIELD OF A POINT CHARGE IN A WEAK GRAVITATIONAL FIELD

It is clear from Eq. (11) that our analysis requires vertical components of forces due to the electric field at each charge. The magnetic field does not enter since the charges are at rest. Of course, if one neglects the effects of gravity upon the electromagnetic field, then each charge produces a pure Coulomb field which leads to a pure repulsion of the particles and no vertical components of force of one on the other. In order to complete our analysis, we must go beyond the Coulomb approximation and determine the distortion of the electrostatic field of a point charge in the presence of a weak gravitational field. Our analysis through first order in g follows from the principle of equivalence. A full treatment of the interplay between electromagnetism and gravitation requires the general theory of relativity.[2]

The effects of a static homogeneous gravitational field $\mathbf{g}$ can be obtained by going to a coordinate frame accelerating with acceleration $\mathbf{g}$ which is free of gravitational fields. Thus for our problem of two point charges supported by strings against a downwards gravitational field, we would find in an accelerating frame that the point charges are accelerating upwards due to external forces from the two strings. Now the electromagnetic forces of each particle upon the other for two particles accelerating side by side were determined earlier in the article[1] on the inertial mass of a system of two point charges. If one simply replaces the acceleration a of that article by the value g, then the previous analysis can be used when specialized to velocity $\mathbf{v} = 0$ so that the particles are observed when instantaneously at rest.

However, the electric field of a point charge in a weak gravitational field is an interesting quantity in its own right. Moreover a sketch of field lines such as given in Fig. 1 provides a qualitative sense of how the electric force of one charge upon the other will lead to an increase in the weight of a system of two point charges. Hence we will go beyond the calculation given for the inertial mass example[1] and will determine the electric field of a point charge in a weak gravitational field.

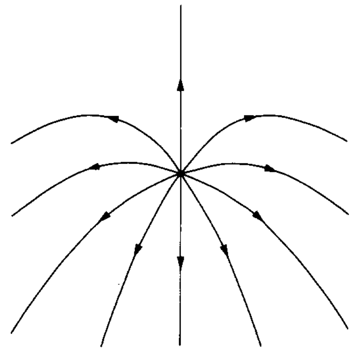

Fig. 1. Electric field of a point charge supported at rest against a weak downwards gravitational field.

From the considerations above regarding the principle of equivalence, we see that the electromagnetic fields of a point charge e supported at rest in a static homogeneous gravitational field $\mathbf{g}$ are the same as those of a point charge undergoing uniform acceleration $-\mathbf{g}$ upwards and instantaneously at rest. We choose our coordinate system so that $\hat{i}$ is the upwards direction and the particle coordinates are $x_e = (1/2)gt^2$, $y_e = 0$, $z_e = 0$ with the particle at rest at the origin at time $t = 0$. There is azymuthal symmetry about the x axis so we will evaluate the electric and magnetic fields in the x-y plane at $(x,y,0)$ at time $t = 0$.

The general expressions for the electromagnetic fields[3] of a point charge e are

$$\mathbf{E} = e\left(\frac{(\hat{n} - \beta)(1 - \beta^2)}{(1 - \hat{n}\cdot\beta)^3 R^2}\right)_{t_{\text{ret}}} + \frac{e}{c}\left(\frac{\hat{n} \times [(\hat{n} - \beta) \times \dot{\beta}]}{(1 - \hat{n}\cdot\beta)^3 R}\right)_{t_{\text{ret}}}, \tag{12}$$

$$\mathbf{B} = \hat{n}_{\text{ret}} \times \mathbf{E}, \tag{13}$$

where $\hat{n}$ is the unit vector pointing from source point to field point, β is the particle velocity divided by c, and R is the distance from the source point to the field point. The retarded time t_{ret} is such that emission of a light signal from the particle will reach the field point $(x,y,0)$ at time $t = 0$. Setting $\Delta t = 0 - t_{\text{ret}}$, we have

$$c^2(\Delta t)^2 = [x - (1/2)g\,(\Delta t)^2]^2 + y^2. \tag{14}$$

We now carry out a power series expansion in the gravitational field strength g, assuming g a small number. Thus $\Delta t = \Delta t_0 + \Delta t_1 + \cdots$, where Δt_0 is zeroth order in g, Δt_1 first order in g, etc. The equation for the retarded time becomes

$$\begin{aligned} &c^2(\Delta t_0 + \Delta t_1 + \cdots)^2 \\ &\quad = [x - (1/2)g(\Delta t_0 + \Delta t_1 + \cdots)^2]^2 + y^2, \end{aligned} \tag{15}$$

with the separation of terms in g leading to: 0th order

$$c^2(\Delta t_0)^2 = x^2 + y^2, \tag{16}$$

1st order

$$2c^2\,\Delta t_0\,\Delta t_1 = -\,xg(\Delta t_0)^2. \tag{17}$$

Now solving successively for Δt_0 and Δt_1, we find through first order in g,

$$\Delta t = [(x^2 + y^2)^{1/2}/c]\,(1 - xg/2c^2). \tag{18}$$

Next we need to evaluate $\hat{n}$, β, $\dot{\beta}$, and R for the retarded time through first order in g,

$$\hat{n} = \hat{i}\,\frac{[x - (1/2)\,g(\Delta t)^2]}{c\Delta t} + \hat{j}\,\frac{y}{c\Delta t}$$
$$= \hat{i}\left[\frac{1}{(x^2+y^2)^{1/2}}\left(x - \frac{y^2 g}{2c^2}\right)\right] + \hat{j}\left[\frac{1}{(x^2+y^2)^{1/2}}\left(y + \frac{xyg}{2c^2}\right)\right], \tag{19}$$

$$\beta = \frac{-g\Delta t}{c}\,\hat{i} = -\frac{(x^2+y^2)^{1/2}\,g}{c^2}\,\hat{i}, \tag{20}$$

$$\dot{\beta} = \frac{g}{c}\,\hat{i}, \tag{21}$$

$$R = c\Delta t = (x^2+y^2)^{1/2}\left(1 - \frac{xg}{2c^2}\right). \tag{22}$$

Substituting into Eq. (12) and retaining terms through only first order in g, we find

$$\mathbf{E}(x,y,0,t) = \frac{e}{(x^2+y^2)^{3/2}}\left[\hat{i}\left(x - \frac{x^2 g}{c^2} - \frac{y^2 g}{2c^2}\right) + \hat{j}\left(y - \frac{xyg}{2c^2}\right)\right]. \tag{23}$$

Also, the magnetic field follows from Eq. (13) and vanishes through first order in g.

At this point it is easy to lift the restriction of the field to the x-y plane. We conclude that for a point charge e at the origin supported against a weak static homogeneous gravitational field $\mathbf{g} = -g\hat{i}$, the magnetic field vanishes and the electric field is given by[4]

$$\mathbf{E}(x,y,z,t) = \frac{e}{(x^2+y^2+z^2)^{3/2}}\left[\hat{i}\left(x - \frac{x^2 g}{c^2} - \frac{y^2 g}{2c^2} - \frac{z^2 g}{2c^2}\right) + \hat{j}\left(y - \frac{xyg}{2c^2}\right) + \hat{k}\left(z - \frac{xzg}{2c^2}\right)\right]. \tag{24}$$

The expression was obtained through first order in g and hence is valid only near the point charge where the distortion from the Coulomb expression is small. Figure 1 gives a two-dimensional sketch of the field pattern from Eq. (23). The electric field is everywhere tangent to the lines shown.

IV. ELECTRIC FORCES LEADING TO A CHANGE IN SYSTEM WEIGHT

The result we have obtained for the field of a point charge in a gravitational field suggests immediately the basis for a change in weight when two point charges are near each other. If a second positive point charge were located along a horizontal axis through the center of Fig. 1, then the electric field of the first charge would be such as to push the second charge downwards. Hence any supporting string on the second charge must exert a greater upwards force.

Specifically for our situation of two point charges q_1 and q_2 separated from each other horizontally by a distance l, we have forces due to the electric fields from Eq. (23) with $x = 0, y = \pm l$,

$$q_1\mathbf{E}_2 = (q_1 q_2/l^3)\,[\hat{i}(-l^2 g/2c^2) + \hat{j}l], \tag{25}$$

$$q_2\mathbf{E}_1 = (q_2 q_1/l^3)\,[\hat{i}(-l^2 g/2c^2) + \hat{j}(-l)]. \tag{26}$$

Hence the additional external supporting force needed for Eq. (10) can be found by substituting in Eqs. (25) and (26). We obtain

$$\Delta\mathbf{F}^{(g)}_{\text{ext tot}} = -q_1\mathbf{E}_2^{(g)} - q_2\mathbf{E}_1^{(g)} = (q_1 q_2/lc^2)g\hat{i}, \tag{27}$$

precisely as required in Eq. (11) where $q_1 q_2/(lc^2)$ is regarded as a change in the system gravitational mass.

All works out perfectly. The mechanism for the increase in gravitational mass associated with the electrostatic potential energy of two point charges can be understood as the electric forces of each charge upon the other, the electric forces arising from gravitational distortions of the Coulomb field.

V. ADDITIONAL COMMENTS

It has been suggested to the writer that a reader would appreciate comments in connection with Newton's third law and estimates of the magnitude of the gravitational distortion for the Coulomb field. We know that two point charges at rest in free space exert radial forces upon each other which satisfy Newton's third law. However, this is not the case in the presence of a gravitational field. In our example, Eqs. (25) and (26) give the force of each charge upon the other. The horizontal $\hat{j}$ components are indeed equal in magnitude and opposite in direction, but the vertical $\hat{i}$ components are in the same direction. Newton's third law is violated by terms proportional to the gravitational field g.

The distortion of the Coulomb field is easily seen to be unmeasurably small for ordinary conditions on earth. From Eq. (24), we can estimate the fractional distortion of the electric field as given by lg/c^2 where l is a typical length in the problem. Clearly for fixed gravitational field g, the distortion is largest when the length l is largest. For the value of g at the surface of the earth $g \cong 10\ \text{m/sec}^2$ and a length $l \cong 1$ m, the fractional distortion is given by $lg/c^2 \cong 10^{-16}$. If we go to atomic systems where g is taken as the gravitational field due to the nucleus and l is the radius of the atom, the effect is still smaller by far, $lg/c^2 \cong GM/rc^2 \cong 10^{-44}$ where M is the proton mass $M = 1.67 \times 10^{-27}$ kg and $r = 1$ Å. If we go in the opposite direction toward astrophysical systems, the effect is enhanced. However, even if the example of two hanging pith balls with a separation $l \cong 1$ m were performed on a neutron star of one solar mass and 1 km radius where $g \cong 10^{14}\ \text{m/sec}^2$ the fractional distortion is still only 10^{-3}. Clearly our example may be instructive from a theoretical point of view, but it does not lend itself to easy experimental measurement.

[1] T. H. Boyer, Am. J. Phys. **46**, 383 (1978).

[2] See, for example, V. Fock, *The Theory of Space, Time and Gravitation* (Pergamon, New York 1959), Chap. IV.

[3] J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1962), p. 657, Eqs. (14.13) and (14.14).

[4] The electric and magnetic fields of a point charge in a uniform gravitational field are obtained in a different form from a general relativistic analysis by F. Rohrlich, Ann. Phys. **22**, 169 (1963). The radial component of the electric field is also given in F. Rohrlich's book, *Classical Charge Particles* (Addison-Wesley, Reading, MA, 1965), Sec. 8.3, p. 219. In the limit of a weak gravitational field, the radial component obtained from our Eq. (24) is in agreement with that obtained from Rohrlich's book, Eq. (8.23).