

# Mass renormalization in classical electrodynamics

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The electromagnetic mass of a charged object can be calculated by three independent methods: (1)  $m_u = u/c^2$ , where  $u$  is the field energy of the object at rest, (2)  $m_p = p/v$ , where  $p$  is the field momentum when the particle is moving at speed  $v$ , and (3)  $m_s = F/a$ , where  $F$  is the self-force when the object has acceleration  $a$ . In the context of a simple dumbbell model we demonstrate that  $m_p = m_s$ , but these in general exceed  $m_u$ .

## I. INTRODUCTION

In classical electrodynamics the rest mass of a charged particle is greater than that of its uncharged twin; its total mass is the sum of its "mechanical" mass and its "electromagnetic" mass

$$m = m_{\text{mech}} + m_{\text{em}}. \quad (1)$$

The latter term reflects the inertia of the particle's electromagnetic field. There are at least three ways<sup>1</sup> that one might go about calculating this electromagnetic mass for a particle of specified size, shape, and charge: (1) With the particle at rest, find the electrostatic energy ( $u$ ) and divide by  $c^2$ :

$$m_u = u/c^2. \quad (2)$$

(Alternatively, with the particle in motion, we would compute the total electromagnetic energy and divide by  $\gamma c^2$ .) We shall call  $m_u$  the "energy-derived mass" of the particle. (2) With the particle moving at constant speed  $v$ , find the electromagnetic momentum  $p$  and divide by  $v$  (or, for high velocities, by  $\gamma v$ ):

$$m_p = p/v. \quad (3)$$

We shall call  $m_p$  the "momentum-derived mass." (3) Recognizing that, from a dynamical point of view, the electromagnetic inertia is due to the force exerted on the particle by its own fields, we evaluate this "self-force"  $F_s$ , pick out the piece proportional to acceleration (call this piece  $F$ ), and divide by  $a$ :

$$m_s = F/a. \quad (4)$$

(If the particle is moving at high speed, in rectilinear motion,<sup>2</sup> divide by  $\gamma^3 a$ .) We call this the "self-force-derived mass."

The first method exploits Einstein's formula  $E = mc^2$  (or, for high speeds,  $E = \gamma mc^2$ ); the second uses the equation for momentum,  $p = mv$  (or, for high speeds,  $p = \gamma mv$ ); the third relies on Newton's second law  $F = ma$  (or, for high speeds,  $F = \gamma^3 ma$ ). Offhand, one would certainly expect the three approaches to yield the same answer.<sup>3</sup> But it is a notorious paradox, discovered around the turn of the century,<sup>4</sup> that  $m_u$  and  $m_p$  do not in general agree (for a spherical charge distribution they differ by a factor of 4/3)—not, at any rate, if we calculate  $u$  and  $p$  by the standard rules to be found in any textbook on electromagnetism. Less well known is the fact that  $m_s$  agrees with  $m_p$ :

$$m_s = m_p \neq m_u. \quad (5)$$

Poincaré<sup>5</sup> attributed the discrepancy to an incompleteness in the model: a charged particle requires the intervention of some nonelectromagnetic force to hold it together—

as we "turn on" the charge, we must at the same time turn on this so-called "Poincaré stress" to keep the particle from flying apart. Now this "other" force, whatever its physical nature, will also contribute to the energy, the momentum, and hence the mass, of the particle. Poincaré showed that when such stabilizing forces are included the anomaly disappears.

But Poincaré's resolution has not enjoyed universal acceptance, and over the years a number of authors, from Fermi<sup>6</sup> to Rohrlich,<sup>7</sup> have looked for an answer within electrodynamics itself. They point out that the standard method for calculating electromagnetic energy and momentum, which amounts to integrating the first four elements of the stress tensor  $T^{\mu\nu}$  over all space,

$$p^\mu = \int T^{\mu 0} dV, \quad (6)$$

yields a 4-vector only when  $T^{\mu\nu}$  is divergenceless. At the location of a charge, the electromagnetic stress tensor is *not* divergenceless, and so Eq. (6) cannot be regarded as a proper definition of electromagnetic energy and momentum. ( $T^{\mu 0}$  represents the energy and momentum density only in charge-free regions; it is fine for studying radiation at large distances from the source, but unacceptable for calculating the total energy and momentum of a particle.) These authors typically retain  $\int T^{00} dV$  to define the electrostatic energy in the particle's rest frame, and derive the energy and momentum in any other frame by Lorentz transformation.

In spite of much argument,<sup>8</sup> there is no real incompatibility between Poincaré's resolution and Rohrlich's. The instability of a purely electromagnetic particle and the non-conversation of the stress tensor are in fact two sides of the same coin. Poincaré adds a nonelectromagnetic term to make  $T^{\mu\nu}$  divergenceless, Rohrlich sticks with the electromagnetic part and modifies the procedure for calculating momentum. If we insist on defining a electromagnetic energy-momentum 4-vector, then Rohrlich's is the remedy of choice; if we are prepared to forego this, in favor of the *total* energy-momentum 4-vector, and we are willing to include the nonelectromagnetic forces which must, after all, be present to stabilize the particle, then Poincaré's is the appropriate prescription.

Neither of them, however, has anything to say about the self-force-derived mass. Why should  $m_s$  agree with  $m_p$ , and not with  $m_u$ ? The answer is far from obvious, though it is suggestive that  $m_p$  and  $m_s$  can be given a purely classical formulation, whereas  $m_u$  (depending as it does on  $E = mc^2$ ) is explicitly relativistic. If, following Rohrlich, we modify  $p$  to bring  $m_p$  in line with  $m_u$ ,  $m_s$  is unaffected, and the paradox remains. If, on the other hand, we introduce Poincaré

stress, it is unclear what this will do to  $m_s$ .

Our purpose in this paper is to illuminate these issues by studying the simplest possible extended charge distribution: a dumbbell, consisting of two point charges a short distance apart. The virtue of the dumbbell model<sup>9</sup> is that the calculations are, for the most part, transparent. Moreover, since an arbitrary distribution may be assembled out of interacting pairs of point charges, one can easily generalize the dumbbell results to obtain the electromagnetic mass of a spherical shell, or any other configuration of interest. In Sec. II we calculate the energy-derived mass of a dumbbell, and in Sec. III we work out its momentum-derived mass. Section IV is devoted to the self-force-derived mass of the dumbbell. In Sec. V we exploit these results to determine  $m_u$ ,  $m_p$ , and  $m_s$  for a spherical shell, and conclude with some general observations regarding the three electromagnetic masses.<sup>10</sup>

## II. ENERGY-DERIVED MASS OF A DUMBELL

The dumbbell at rest consists of two equal charges,  $q_1 = q_2 = e/2$ , a distance  $d$  apart. The electrostatic energy of such a configuration (in Gaussian units) is

$$u = q_1 q_2 / (|\mathbf{r}_1 - \mathbf{r}_2|) = e^2 / 4d, \quad (7)$$

and hence

$$m_u = e^2 / 4dc^2. \quad (8)$$

This result can also be obtained from the general formula<sup>11</sup>

$$u_{\text{total}} = \frac{1}{8\pi} \int E^2 dV. \quad (9)$$

The dumbbell's electric field consists of two parts, due to  $q_1$  and  $q_2$ , respectively:

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2, \quad (10)$$

where (see Fig. 1)

$$\mathbf{E}_1 = (q_1/r_1^3)\mathbf{r}_1, \quad \mathbf{E}_2 = (q_2/r_2^3)\mathbf{r}_2. \quad (11)$$

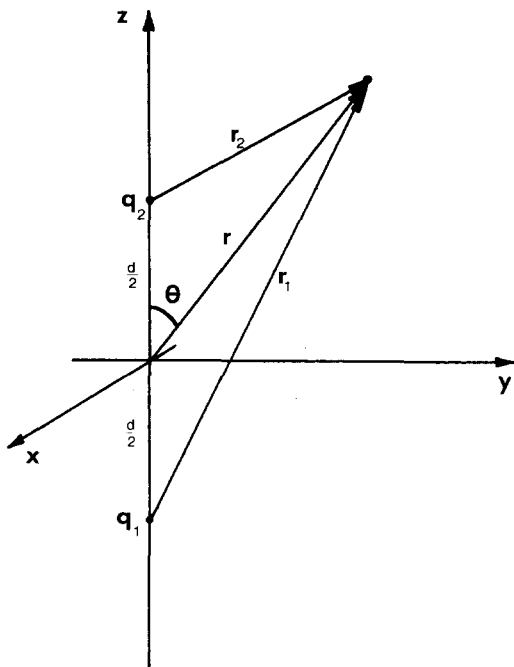


Fig. 1. The dumbbell. Diagram defines the various quantities in Eq. (11).

Thus

$$E^2 = E_1^2 + E_2^2 + 2\mathbf{E}_1 \cdot \mathbf{E}_2. \quad (12)$$

However, we are not concerned here with the first two terms, which correspond to the energy of  $q_1$  alone and  $q_2$  alone (both of which are actually *infinite* for point charges). Rather, we want the *energy of interaction*, which derives from the  $\mathbf{E}_1 \cdot \mathbf{E}_2$  term. So for our purposes

$$u = \frac{1}{4\pi} \int \mathbf{E}_1 \cdot \mathbf{E}_2 dV. \quad (13)$$

Using spherical coordinates as indicated in Fig. 1, we have

$$\begin{aligned} \mathbf{r}_1 &= (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta + d/2), \\ \mathbf{r}_2 &= (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta - d/2), \end{aligned} \quad (14)$$

$$dV = r^2 \sin \theta dr d\theta d\phi,$$

and hence

$$\begin{aligned} u &= \frac{e^2}{16\pi} \int \frac{r^2 - d^2/4}{[(r^2 + d^2/4)^2 - (rd \cos \theta)^2]^{3/2}} \\ &\quad \times r^2 \sin \theta dr d\theta d\phi. \end{aligned} \quad (15)$$

Integrating  $\phi$  (from 0 to  $2\pi$ ),  $\theta$  (from 0 to  $\pi$ ), and  $r$  (from 0 to  $\infty$ ), we recover Eq. (7).

## III. MOMENTUM-DERIVED MASS OF A DUMBELL

If the dumbbell is now set in motion, at constant velocity  $\mathbf{v}$ , its fields carry momentum<sup>12</sup> as well as energy:

$$\mathbf{p} = \frac{1}{4\pi c} \int (\mathbf{E} \times \mathbf{B}) dV. \quad (16)$$

Here  $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$  and  $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2$ , so that

$$\begin{aligned} \mathbf{E} \times \mathbf{B} &= (\mathbf{E}_1 \times \mathbf{B}_1) + (\mathbf{E}_2 \times \mathbf{B}_2) \\ &\quad + (\mathbf{E}_1 \times \mathbf{B}_2) + (\mathbf{E}_2 \times \mathbf{B}_1). \end{aligned} \quad (17)$$

As before, we are not interested in the first two terms, which correspond to the (infinite) momentum of either end alone—it is the *interaction momentum*  $(\mathbf{E}_1 \times \mathbf{B}_2) + (\mathbf{E}_2 \times \mathbf{B}_1)$  that concerns us here. Now, the magnetic field of a point charge in motion is given by<sup>13</sup>

$$\mathbf{B} = (\mathbf{v}/c) \times \mathbf{E}, \quad (18)$$

so

$$\begin{aligned} \mathbf{E}_1 \times \mathbf{B}_2 &= (1/c) \mathbf{E}_1 \times (\mathbf{v} \times \mathbf{E}_2) \\ &= (1/c) [\mathbf{v}(\mathbf{E}_1 \cdot \mathbf{E}_2) - \mathbf{E}_2(\mathbf{v} \cdot \mathbf{E}_1)]. \end{aligned} \quad (19)$$

Thus for our purposes

$$\begin{aligned} \mathbf{p} &= \frac{1}{4\pi c^2} \int [2\mathbf{v}(\mathbf{E}_1 \cdot \mathbf{E}_2) - \mathbf{E}_1(\mathbf{v} \cdot \mathbf{E}_2) \\ &\quad - \mathbf{E}_2(\mathbf{v} \cdot \mathbf{E}_1)] dV. \end{aligned} \quad (20)$$

For nonrelativistic velocities ( $v \ll c$ ) the electric fields are just the Coulomb fields of Eq. (11). (There are relativistic corrections,<sup>14</sup> but they are second order in  $v/c$ .) Accordingly,

$$\begin{aligned} \mathbf{p} &= \frac{e^2}{16\pi c^2} \int \frac{1}{(r_1 r_2)^3} [2\mathbf{v}(\mathbf{r}_1 \cdot \mathbf{r}_2) \\ &\quad - \mathbf{r}_1(\mathbf{v} \cdot \mathbf{r}_2) - \mathbf{r}_2(\mathbf{v} \cdot \mathbf{r}_1)] dV, \end{aligned} \quad (21)$$

where  $\mathbf{r}_1$ ,  $\mathbf{r}_2$ , and  $dV$  are given by Eq. (14).

### A. Longitudinal motion

If  $\mathbf{v}$  points along the axis of the dumbbell ( $\mathbf{v} = v\hat{z}$ , in Fig. 1), Eq. (21) reduces to

$$\mathbf{p} = \frac{e^2 \mathbf{v}}{8\pi c^2} \int \frac{r^2 \sin^2 \theta}{[(r^2 + d^2/4)^2 - (rd \cos \theta)^2]^{3/2}} \times r^2 \sin \theta dr d\theta d\phi, \quad (22)$$

and we obtain, upon performing the integrals,

$$\mathbf{p} = (e^2/2c^2 d) \mathbf{v}. \quad (23)$$

The corresponding momentum-derived mass is

$$m_p = e^2/2c^2 d, \quad (24)$$

which is *twice* the energy-derived mass.

### B. Transverse motion

If the velocity is perpendicular to the dumbbell axis (we may as well take  $\mathbf{v} = v\hat{x}$ , in Fig. 1), Eq. (21) yields

$$\mathbf{p} = \frac{e^2 \mathbf{v}}{8\pi c^2} \int \frac{r^2(1 - \sin^2 \theta \cos^2 \phi) - d^2/4}{[(r^2 + d^2/4)^2 - (rd \cos \theta)^2]^{3/2}} \times r^2 \sin \theta dr d\theta d\phi, \quad (25)$$

leading to

$$\mathbf{p} = (e^2/4c^2 d) \mathbf{v}, \quad (26)$$

and hence

$$m_p = e^2/4c^2 d. \quad (27)$$

Evidently the momentum-derived mass depends on the direction of the velocity, and for transverse motion it agrees with the energy-derived mass.

### C. Arbitrary motion

If  $\mathbf{v}$  makes an angle  $\psi$  with the dumbbell axis [ $\mathbf{v} = v(\sin \psi \hat{x} + \cos \psi \hat{z})$  in Fig. 1], Eq. (21) gives

$$\mathbf{p} = \frac{e^2 v}{8\pi c^2} \left( \sin \psi \hat{x} \int \frac{r^2(1 - \sin^2 \theta \cos^2 \phi) - d^2/4}{(r_1 r_2)^3} dV + \cos \psi \hat{z} \int \frac{r^2 \sin^2 \theta}{(r_1 r_2)^3} dV \right), \quad (28)$$

and hence

$$\mathbf{p} = (e^2 v/4c^2 d)(\sin \psi \hat{x} + 2 \cos \psi \hat{z}), \quad (29)$$

or, letting  $\hat{d}$  be a unit vector along the dumbbell axis

$$\mathbf{p} = (e^2/4c^2 d)[\mathbf{v} + (\mathbf{v} \cdot \hat{d})\hat{d}]. \quad (30)$$

The momentum is not in general parallel to the velocity; the components parallel and perpendicular to  $\mathbf{v}$  are

$$p_{\parallel} = (e^2 v/4c^2 d)(1 + \cos^2 \psi); \quad (31)$$

$$p_{\perp} = (e^2 v/4c^2 d) \sin \psi \cos \psi,$$

so it is only for longitudinal and transverse motion that the notion of momentum-derived mass carries unambiguous meaning.

## IV. SELF-FORCE-DERIVED MASS OF A DUMBBELL

When the dumbbell *accelerates*, the electromagnetic force of  $q_1$  on  $q_2$  is not equal and opposite to the force of  $q_2$  on  $q_1$ ; there is a net force *of the dumbbell on itself*, which we call  $\mathbf{F}_{\text{self}}$ . Our purpose in this section is to calculate  $\mathbf{F}_{\text{self}}$ ,

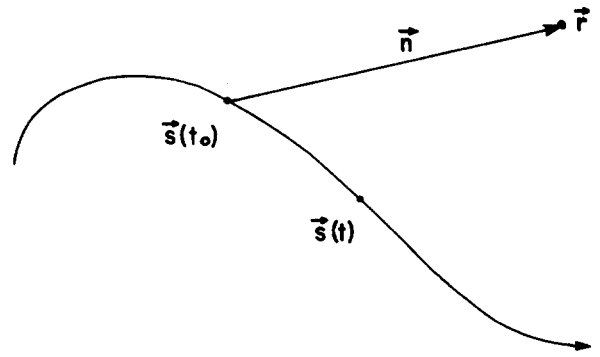


Fig. 2. Retarded position. Fields at  $\mathbf{r}$  are determined by the status of the charge at the retarded time  $t_0$ , when it was at  $\mathbf{s}(t_0)$ .

and obtain from it the “self-force-derived mass,”  $m_s$ . Now, the electric field of a charge  $q$  in arbitrary motion is given by<sup>15</sup>

$$\mathbf{E}(\mathbf{r}, t) = [qn/(\mathbf{n} \cdot \mathbf{m})^3][(c^2 - v^2)\mathbf{m} + \mathbf{n} \times (\mathbf{m} \times \mathbf{a})], \quad (32)$$

where  $\mathbf{v}$  (the particle’s velocity),  $\mathbf{a}$  (its acceleration),  $\mathbf{s}$  (its position),  $\mathbf{n} = (\mathbf{r} - \mathbf{s})$ , and  $\mathbf{m} = (c\hat{n} - \mathbf{v})$ , are all to be evaluated at the retarded time  $t_0$ , defined implicitly by

$$c(t - t_0) = |\mathbf{r} - \mathbf{s}(t_0)| = n \quad (33)$$

(see Fig. 2). The magnetic field, meanwhile, is  $\mathbf{B}(\mathbf{r}, t) = \hat{n} \times \mathbf{E}$ .

### A. Longitudinal motion

Suppose the dumbbell is constrained to move along the line of its axis. Let  $\mathbf{E}_1(t)$  be the electric field of  $q_1$  at the location of  $q_2$ , and  $\mathbf{E}_2(t)$  the field of  $q_2$  at  $q_1$ . In this case  $\mathbf{n}$ ,  $\mathbf{m}$ ,  $\mathbf{v}$ , and  $\mathbf{a}$ , all lie along the  $z$  axis, so that the second term in (32) vanishes, and

$$\hat{n}_{1,2} = \pm \hat{z}, \quad \mathbf{m}_{1,2} = (\pm c - v)\hat{z}.$$

Meanwhile, from (33),

$$n_{1,2} = c(t - t_{1,2}),$$

where  $t_1$  and  $t_2$  are the respective retarded times. Thus Eq. (32) reduces to

$$\mathbf{E}_1(t) = \frac{e}{2c^2(t - t_1)^2} \left( \frac{c + v_1(t_1)}{c - v_1(t_1)} \right) \hat{z}, \quad (34)$$

$$\mathbf{E}_2(t) = - \frac{e}{2c^2(t - t_2)^2} \left( \frac{c - v_2(t_2)}{c + v_2(t_2)} \right) \hat{z},$$

where  $v_1(t_1)$  is the velocity of end (1) at time  $t_1$ , and  $v_2(t_2)$  is the velocity of end (2) at time  $t_2$ . If  $z(t)$  is the position of the center of the dumbbell at time  $t$ , then the ends are at  $z(t) \pm d/2$ , so the retarded time condition (33) becomes

$$c(t - t_1) = |[z(t) + d/2] - [z(t_1) - d/2]| = z(t) - z(t_1) + d, \quad (35)$$

$$c(t - t_2) = |[z(t) - d/2] - [z(t_2) + d/2]| = z(t_2) - z(t) + d.$$

In writing Eq. (35) we assume the dumbbell is rigid *in the lab frame*, with a fixed length  $d$ . The model is to this extent nonrelativistic, since it makes reference to a particular inertial frame. A fully relativistic treatment is given in the Appendix. Note that the magnetic field on the axis is zero. The

self-force on the dumbbell is therefore

$$\mathbf{F}_{\text{self}} = (e/2)(\mathbf{E}_1 + \mathbf{E}_2). \quad (36)$$

Unfortunately, the retarded time condition [Eq. (35)] cannot be solved in closed form. However, it is easy to obtain the general solution as a power series in  $d$ :

$$t_1 = t - \frac{d}{(c-v)} + \frac{ad^2}{2(c-v)^3} + (\dots)d^3 + \dots, \quad (37)$$

$$t_2 = t - \frac{d}{(c+v)} - \frac{ad^2}{2(c+v)^3} + (\dots)d^3 + \dots,$$

where  $v$  and  $a$  are the velocity and acceleration of the center at time  $t$ :  $v = dz(t)/dt$ ,  $a = d^2z(t)/dt^2$ . This leads to a corresponding expansion of the fields

$$\mathbf{E}_1 = \frac{e}{2c^2} \left( \frac{(c^2 - v^2)}{d^2} - \frac{a}{d} + (\dots)d^0 + (\dots)d + \dots \right) \hat{\mathbf{z}}, \quad (38)$$

$$\mathbf{E}_2 = -\frac{e}{2c^2} \left( \frac{(c^2 - v^2)}{d^2} + \frac{a}{d} + (\dots)d^0 + (\dots)d + \dots \right) \hat{\mathbf{z}}.$$

The self-force on the dumbbell is therefore

$$\mathbf{F}_{\text{self}} = [ - (e^2/2c^2d)a + (\dots)d^0 + (\dots)d + \dots ] \hat{\mathbf{z}}. \quad (39)$$

The leading term here is proportional to the acceleration<sup>16</sup>; in Newton's second law

$$\mathbf{F}_{\text{external}} + \mathbf{F}_{\text{self}} = m_{\text{mech}} \mathbf{a}, \quad (40)$$

it is natural to pull this term over to the right-hand side, where it has the effect of adding to the mass an amount

$$(E_1)_x = \frac{e}{2} c \left( \frac{[c^2 - v(t_1)^2][x(t) - x(t_1) - (t - t_1)v(t_1)] - d^2 a(t_1)}{\{c^2(t - t_1) - [x(t) - x(t_1)]v(t_1)\}^3} \right). \quad (42)$$

In this case the retarded time condition (33) reduces to

$$c^2(t - t_1)^2 = [x(t) - x(t_1)]^2 + d^2. \quad (43)$$

The net self-force on the dumbbell is then

$$\mathbf{F}_{\text{self}} = 2(e/2)(E_1)_x \hat{\mathbf{x}}. \quad (44)$$

As before, we solve the retarded time condition [Eq. (43)] in the form of a power series in  $d$ :

$$t_1 = t - \frac{1}{\sqrt{c^2 - v^2}} d + \frac{va}{2(c^2 - v^2)} d^2 + (\dots)d^3 + \dots, \quad (45)$$

where  $v$  and  $a$  are the velocity and acceleration of the center at time  $t$ :  $v = dx(t)/dt$ ,  $a = d^2x(t)/dt^2$ . The resulting expansion of  $(E_1)_x$  is

$$(E_1)_x = -\frac{e}{2} c \frac{a}{2(c^2 - v^2)^{3/2}} \frac{1}{d} + (\dots)d^0 + (\dots)d^2 + \dots, \quad (46)$$

and the self-force on the dumbbell is<sup>16</sup>

$$\mathbf{F}_{\text{self}} = [ -\gamma^3(e^2/4c^2d)a + (\dots)d^0 + (\dots)d + \dots ] \hat{\mathbf{x}}. \quad (47)$$

Evidently the self-force-derived mass for the case of transverse motion is

$$m_s = \gamma^3(e^2/4c^2d). \quad (48)$$

All of our other calculations have been nonrelativistic, and it comes as a surprise to see the correct relativistic factor of  $\gamma^3$  emerge at this stage. But the reason is simple<sup>18</sup>:

$$m_s = e^2/2c^2d. \quad (41)$$

Evidently, for longitudinal motion the self-force-derived mass agrees with the momentum-derived mass [Eq. (24)], but not the energy-derived mass [Eq. (8)].

## B. Transverse motion

Suppose now that the motion is *perpendicular* to the axis of the dumbbell<sup>17</sup>—say,  $v$  is along the  $x$  direction in Fig. 1. The magnetic forces are in the  $z$  direction, and they cancel. By symmetry, the net self-force is in the  $x$  direction, and the two ends contribute equally, so we need only calculate the  $x$  component of the electric field at  $q_2$ , due to  $q_1$ . If  $x(t)$  is the position of the center of the dumbbell, at time  $t$ , then end (2) is at

$$\mathbf{r}(t) = x(t)\hat{\mathbf{x}} + d/2\hat{\mathbf{z}},$$

while, at the retarded time  $t_1$ , end (1) was at

$$\mathbf{s}(t_1) = x(t_1)\hat{\mathbf{x}} - d/2\hat{\mathbf{z}}.$$

So the terms in Eq. (32) are

$$\mathbf{n} = [x(t) - x(t_1)]\hat{\mathbf{x}} + d\hat{\mathbf{z}},$$

$$n = c(t - t_1) = \{[x(t) - x(t_1)]^2 + d^2\}^{1/2},$$

$$\mathbf{v} = v(t_1)\hat{\mathbf{x}},$$

$$\mathbf{a} = a(t_1)\hat{\mathbf{x}},$$

$$\mathbf{m} = \frac{[x(t) - x(t_1) - (t - t_1)v(t_1)]\hat{\mathbf{x}} + d\hat{\mathbf{z}}}{(t - t_1)},$$

$$\mathbf{n} \cdot \mathbf{m} = c^2(t - t_1) - [x(t) - x(t_1)]v(t_1),$$

and hence

The transverse dumbbell is in fact a fully relativistic model, whereas the longitudinal dumbbell, as presented here, is not. For in the longitudinal case we did not take into account the Lorentz contraction of the dumbbell, while in the transverse case there is no contraction. In the Appendix we develop the relativistic version of the dumbbell in longitudinal motion, and the appropriate factor of  $\gamma^3$  then emerges naturally. For the moment, though, we want the nonrelativistic limit of Eq. (48):

$$m_s = e^2/4c^2d. \quad (49)$$

In transverse motion, then, the energy-derived mass [Eq. (8)], momentum-derived mass [Eq. (27)], and self-force-derived mass [Eq. (49)] are all equal.

## C. Arbitrary motion

Finally, suppose the motion is along a line which makes an angle  $\psi$  with the dumbbell axis, so that the center is at

$$\mathbf{s}(t) = r(t)(\sin \psi \hat{\mathbf{x}} + \cos \psi \hat{\mathbf{z}}) \quad (50)$$

in Fig. 1 and the two ends are at

$$\mathbf{s}_1(t) = r(t) \sin \psi \hat{\mathbf{x}} + [r(t) \cos \psi - d/2] \hat{\mathbf{z}}, \quad (51)$$

$$\mathbf{s}_2(t) = r(t) \sin \psi \hat{\mathbf{x}} + [r(t) \cos \psi + d/2] \hat{\mathbf{z}}.$$

The retarded times  $t_1$  and  $t_2$  are determined by the condi-

tions

$$\begin{aligned} c^2(t-t_1)^2 &= [\mathbf{s}_2(t) - \mathbf{s}_1(t_1)]^2 \\ &= [r(t) - r(t_1)]^2 + 2[r(t) - r(t_1)]\cos\psi d + d^2, \\ c^2(t-t_2)^2 &= [\mathbf{s}_1(t) - \mathbf{s}_2(t_2)]^2 \\ &= [r(t) - r(t_2)]^2 - 2[r(t) - r(t_2)]\cos\psi d + d^2. \end{aligned} \quad (52)$$

To simplify the algebra at no real cost, let us assume the dumbbell is instantaneously at rest, at time  $t$ . The series solution to Eq. (52) then takes the form

$$t_1 = t - (1/c)d + (a \cos\psi/2c^3)d^2 + ( )d^3 + \dots, \quad (53)$$

$$t_2 = t - (1/c)d - (a \cos\psi/2c^3)d^2 + ( )d^2 + \dots,$$

where  $a = d^2 r/dt^2$  is the acceleration of the dumbbell at time  $t$ . Expanding Eq. (32) in powers of  $d$ , we obtain

$$\begin{aligned} \mathbf{E}_1(t) &= \frac{e}{2} \left( \frac{1}{d^2} \hat{z} - \frac{a}{2c^2 d} (\sin\psi \hat{x} + 2 \cos\psi \hat{z}) \right. \\ &\quad \left. + ( )d^0 + ( )d^1 + \dots \right), \end{aligned} \quad (54)$$

$$\begin{aligned} \mathbf{E}_2(t) &= \frac{e}{2} \left( -\frac{1}{d^2} \hat{z} - \frac{a}{2c^2 d} (\sin\psi \hat{x} + 2 \cos\psi \hat{z}) \right. \\ &\quad \left. + ( )d^0 + ( )d^1 + \dots \right). \end{aligned}$$

Since the charges are at rest, the magnetic forces are zero, and hence

$$\begin{aligned} \mathbf{F}_{\text{self}} &= (e/2)(\mathbf{E}_1 + \mathbf{E}_2) \\ &= -(e^2 a/4c^2 d)(\sin\psi \hat{x} + 2 \cos\psi \hat{z}) \\ &\quad + ( )d^0 + ( )d^1 + \dots \end{aligned} \quad (55)$$

We are concerned here with the term which is proportional to  $a$  (call it  $\mathbf{F}$ , for short)<sup>16</sup>:

$$\mathbf{F} = -(e^2 a/4c^2 d)(\sin\psi \hat{x} + 2 \cos\psi \hat{z}), \quad (56)$$

or, letting  $\hat{d}$  be a unit vector along the dumbbell axis,

$$\mathbf{F} = -(e^2/4c^2 d)[\mathbf{a} + (\mathbf{a} \cdot \hat{d})\hat{d}]. \quad (57)$$

$\mathbf{F}$  does not in general lie along the direction of motion; the components parallel and perpendicular to  $\mathbf{a}$  are

$$\begin{aligned} F_{\parallel} &= -(e^2 a/4c^2 d)(1 + \cos^2\psi); \\ F_{\perp} &= -(e^2 a/4c^2 d)\sin\psi \cos\psi. \end{aligned} \quad (58)$$

Notice that the dependence of  $F$  on  $a$  is identical to that of  $p$  on  $v$  [Eqs. (29)–(31)]. Only  $F_{\parallel}$  can be regarded as a purely inertial term; accordingly

$$m_s = (e^2/4c^2 d)(1 + \cos^2\psi). \quad (59)$$

## V. ELECTROMAGNETIC MASS OF A SPHERICAL SHELL

Given the formulas for energy, momentum, and self-force of a dumbbell, it is a straightforward matter to generalize to arbitrary rigid charge distributions: we simply chop the configuration up into pairs of infinitesimal charges and integrate (dividing by 2 to avoid counting the same pair twice). For example, suppose we have a spherical shell of radius  $R$  carrying a uniform surface charge density  $\sigma$ . We divide it into patches of area  $da$  (Fig. 3), so that the charge at each end of the “dumbbell” is  $e/2 \rightarrow \sigma da$ .

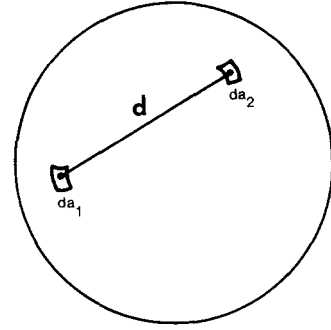


Fig. 3. Spherical shell. Surface is divided into infinitesimal patches, and the dumbbell results are applied to each pair.

### A. Energy

From Eq. (7) we have

$$u = \frac{\sigma^2}{2} \iint \frac{1}{d} da_1 da_2 = \frac{\sigma^2}{2} \iint \left( \int \frac{1}{d} da_1 \right) da_2. \quad (60)$$

We may as well orient our  $x_1, y_1, z_1$  axes so that  $da_2$  lies at the “north pole”; then

$$d = \sqrt{2R} \sqrt{1 - \cos\theta_1}, \quad (61)$$

and hence

$$\int \frac{1}{d} da_1 = \int \frac{R^2 \sin\theta_1}{\sqrt{2R} \sqrt{1 - \cos\theta_1}} d\theta_1 d\phi_1 = 4\pi R, \quad (62)$$

so that

$$u = \frac{\sigma^2}{2} (4\pi R) \int da_2 = 8\pi^2 \sigma^2 R^3. \quad (63)$$

Or, in terms of the total charge  $Q = 4\pi R^2 \sigma$ ,

$$u = \frac{1}{2}(Q^2/R). \quad (64)$$

This familiar result can of course be obtained by simpler means<sup>19</sup>; we present it here only as an illustration of method. Evidently the energy-derived mass of a spherical shell is

$$m_u = Q^2/2Rc^2. \quad (65)$$

### B. Momentum

Suppose now that the sphere is moving at constant speed  $v$  in the  $z$  direction. Although each infinitesimal dumbbell has a component of momentum perpendicular to the velocity, symmetry dictates that these components cancel when we integrate over the sphere. Accordingly, the total momentum of the sphere [quoting Eq. (31)] is

$$\mathbf{p} = \frac{\sigma^2 \mathbf{v}}{2c^2} \iint \frac{(1 + \cos^2\psi)}{d} da_1 da_2, \quad (66)$$

where  $\psi$  is the angle between  $\mathbf{d}$  and  $\mathbf{v}$ :

$$\cos^2\psi = d_z^2/d^2. \quad (67)$$

Now, it is clear from symmetry that

$$\begin{aligned} \iint \frac{d_x^2}{d^3} da_1 da_2 &= \iint \frac{d_y^2}{d^3} da_1 da_2 \\ &= \iint \frac{d_z^2}{d^3} da_1 da_2, \end{aligned} \quad (68)$$

while their sum is simply  $\iint (1/d) da_1 da_2$ . Evidently, then,

$$\iint \frac{\cos^2\psi}{d} da_1 da_2 = \frac{1}{3} \iint \frac{1}{d} da_1 da_2, \quad (69)$$

and hence

$$\mathbf{p} = \frac{4}{3} \frac{\mathbf{v}}{c^2} \frac{\sigma^2}{2} \int \int \frac{1}{d} da_1 da_2 = \frac{4}{3} \frac{\mathbf{v}}{c^2} u. \quad (70)$$

Consequently, the momentum-derived mass of a spherical shell is 4/3 its energy-derived mass<sup>20</sup>:

$$m_p = \frac{4}{3} m_u = \frac{2}{3} \frac{Q^2}{Rc^2}. \quad (71)$$

### C. Self-force

The self-force calculation [starting from Eq. (58)] is identical to that for momentum:

$$\mathbf{F} = -\frac{4}{3} (\mathbf{a}/c^2) u, \quad (72)$$

and so

$$m_s = m_p = \frac{4}{3} m_u. \quad (73)$$

## VI. CONCLUSION

We have calculated the electromagnetic energy, momentum, and self-force for dumbbell and spherical shell configurations. Each result yields a particular expression for the electromagnetic mass of the object; however, the three methods do not agree. Whereas the momentum- and self-force-derived masses are always equal, they exceed the energy-derived mass by an amount which depends on how the charge is distributed along the line of motion:

$$m_s = m_p \geq m_u. \quad (74)$$

Only when the charge is confined to the plane perpendicular to the motion are all three masses in agreement. Moreover, since the momentum is not in general parallel to the velocity, nor the self-force parallel to the acceleration, the corresponding masses lose their scalar character (the sphere is an exception). The discrepancy between  $m_p$  and  $m_u$  is best resolved by Rohrlich's method,<sup>7</sup> which disqualifies our calculation of  $\mathbf{p}$ . But Rohrlich's procedure does not address the problem of self-force-derived mass, and to our knowledge  $m_s \neq m_u$  remains an unsolved paradox.<sup>21</sup>

## APPENDIX: LORENTZ-CONTRACTED DUMBBELL

In calculating the self-force on a dumbbell in longitudinal motion we assumed that it is rigid in the "lab" frame—that is, its length is  $d$  from our point of view. This gives special importance to a particular inertial system, and it is not surprising that the resulting mass [Eq. (41)] is missing the relativistic factor of  $\gamma^3$ . Rigidity is a notoriously slippery concept in relativity, but here we have the advantage that there are only two points to worry about: the two ends of the dumbbell. With this in mind we define the covariant dumbbell as follows: Let  $z(t)$  be the position of the "center," and let  $S'$  be the inertial system in which the center is instantaneously at rest. We require that the dumbbell have a length  $d$  in  $S'$ . Specifically, at time  $t_0$  (in the lab frame,  $S$ ) the coordinates of the center are

$$\text{center: } z_0 = z(t_0), \quad t_0. \quad (75)$$

By Lorentz transformation, the coordinates of the center in  $S'$  are

$$\text{center: } z'_0 = \gamma_0(z_0 - v_0 t_0), \quad t'_0 = \gamma_0[t_0 - (v_0/c^2)z_0], \quad (76)$$

where  $v_0 = (dz/dt)$  at time  $t_0$ , and  $\gamma_0 = 1/\sqrt{1 - v_0^2/c^2}$ . At this instant (in  $S'$ ) the coordinates of the two ends are

$$q_1: \quad z'_1 = z'_0 - d/2, \quad t'_0, \quad (77)$$

$$q_2: \quad z'_2 = z'_0 + d/2, \quad t'_0.$$

Transforming back to  $S$ , the coordinates are

$$q_1: \quad z_1 = \gamma_0(z'_1 + v_0 t'_0) = z_0 - \gamma_0(d/2),$$

$$t_{(1)} = \gamma_0[t'_0 + (v_0/c^2)z'_1] = t_0 - \gamma_0 v_0(d/2c^2), \quad (78)$$

$$q_2: \quad z_2 = \gamma_0(z'_2 + v_0 t'_0) = z_0 + \gamma_0(d/2),$$

$$t_{(2)} = \gamma_0[t'_0 + (v_0/c^2)z'_2] = t_0 + \gamma_0 v_0(d/2c^2). \quad (79)$$

Equation (78) specifies the position ( $z_1$ ) of  $q_1$  at lab time  $t_{(1)}$ . Unfortunately, it is expressed in terms of the parameter  $t_0$ . We would like to eliminate  $t_0$  and write  $z_1$  directly as a function of lab time  $t$ . To begin with we must solve for  $t_0$  in terms of  $t = t_{(1)}$ :

$$t = t_0 - \gamma_0 v_0(d/2c^2). \quad (80)$$

As always, we expand in powers of  $d$ :

$$(t_0 - t) = \frac{d}{2c^2} \left[ \gamma v + (t_0 - t) \frac{d}{dt} (\gamma v) + \dots \right] \quad (81)$$

(unsubscripted quantities are evaluated at time  $t$ ). Now

$$\frac{d}{dt} (\gamma v) = \gamma^3 a, \quad (82)$$

so

$$(t_0 - t) [1 - (d/2c^2)\gamma^3 a + \dots] = (d/2c^2)\gamma v, \quad (83)$$

and hence

$$t_0 = t + (\gamma v/2c^2)d + (\gamma^4 v a/4c^4)d^2 + \dots \quad (84)$$

This gives us  $t_0$  in terms of  $t$ ; returning to Eq. (78) we have

$$z_1(t) = z(t_0) - \gamma_0(d/2). \quad (85)$$

But

$$z(t_0) = z + (t_0 - t)v + \frac{1}{2}(t_0 - t)^2 a + \dots \quad (86)$$

and

$$\gamma_0 = \gamma + (t_0 - t) \frac{d\gamma}{dt} + \dots \quad (87)$$

Putting this together, and using (84), we conclude

$$z_1(t) = z(t) - \frac{1}{\gamma} \frac{d}{2} + \frac{av^2\gamma^2}{8c^4} d^2 + \dots \quad (88)$$

The corresponding formula for the location of  $q_2$  is obtained from (79)—or, more simply, by changing the sign of  $d$  in Eq. (88):

$$z_2(t) = z(t) + \frac{1}{\gamma} \frac{d}{2} + \frac{av^2\gamma^2}{8c^4} d^2 + \dots \quad (89)$$

Notice that  $z_2(t) - z_1(t) = d/\gamma$  to this order, so the dumbbell has the appropriately contracted length in the lab.

The next step is to compute the retarded time  $t_1$ , given by

$$\begin{aligned} c(t - t_1) &= z_2(t) - z_1(t_1) \\ &= [z(t) - z(t_1)] + \left( \frac{1}{\gamma} + \frac{1}{\gamma_1} \right) \frac{d}{2} \\ &\quad + (av^2\gamma^2 - a_1 v_1^2 \gamma_1^2)(d^2/8c^4) + \dots \end{aligned} \quad (90)$$

Now

$$[z(t) - z(t_1)] = (t - t_1)v - \frac{1}{2}(t - t_1)^2 a + \dots,$$

$$\left[ \frac{1}{\gamma} + \frac{1}{\gamma_1} \right] = \frac{1}{\gamma} + \left[ \frac{1}{\gamma} + (t_1 - t) \frac{d}{dt} \left( \frac{1}{\gamma} \right) + \dots \right]$$

$$= \frac{2}{\gamma} + \frac{\gamma v a}{c^2} (t - t_1) + \dots,$$

and

$$(av^2\gamma^2 - a_1v_1^2\gamma_1^2) = (t - t_1) \frac{d}{dt} (av^2\gamma^2) + \dots \quad (91)$$

Since  $t - t_1$  is of order  $d$ , the last term contributes to Eq. (90) only in third order, and can be dropped, leaving

$$c(t - t_1) = v(t - t_1) - \frac{1}{2}a(t - t_1)^2 + \frac{d}{\gamma}$$

$$+ \frac{d\gamma v a}{2c^2} (t - t_1) + \dots \quad (92)$$

This is a quadratic equation for  $(t - t_1)$ ; expanding the solution in powers of  $d$ , we obtain

$$t_1 = t - \frac{1}{\gamma(c - v)} d + \frac{a}{2c(c - v)^2} d^2 + ( ) d^3 + \dots \quad (93)$$

To calculate  $\mathbf{E}_1(t)$ —Eq. (34)—we need to know  $v_1(t_1)$ , the velocity of end (1) at time  $t_1$ . According to (78), the speed of  $q_1$  at any lab time  $t_{(1)}$  is

$$v_1(t_{(1)}) = \frac{dz_1}{dt_{(1)}} = \frac{dz_1}{dt_0} \frac{dt_0}{dt_{(1)}}, \quad (94)$$

while

$$\frac{dz_1}{dt_0} = \frac{dz_0}{dt_0} - \frac{d}{2} \frac{d\gamma_0}{dt_0} = v_0 \left( 1 - \frac{d}{2} \frac{\gamma_0^3 a_0}{c^2} \right), \quad (95)$$

and

$$\frac{dt_{(1)}}{dt_0} = 1 - \frac{d}{2c^2} \frac{d}{dt_0} (\gamma_0 v_0) = \left( 1 - \frac{d}{2} \frac{\gamma_0^3 a_0}{c^2} \right). \quad (96)$$

Thus

$$v_1(t_1) = v(\bar{t}_0). \quad (97)$$

Surprisingly, the velocity of end (1) at time  $t_1$  is the same as the velocity of the center at time  $\bar{t}_0$ , defined implicitly by the equation

$$t_1 = \bar{t}_0 - \bar{\gamma}_0 \bar{v}_0 (d/2c^2). \quad (98)$$

The series solution to Eq. (98) [quoting (84)] is

$$\bar{t}_0 = t_1 + \gamma(t_1)v(t_1)(d/2c^2) + ( ) d^2 + \dots, \quad (99)$$

so

$$v_1(t_1) = v(t_1) + a(t_1) [\gamma(t_1)v(t_1)/2c^2] d + ( ) d^2 + \dots \quad (100)$$

But

$$v(t_1) = v + (t_1 - t)a + \dots = v - [a/\gamma(c - v)]d + \dots, \quad (101)$$

and hence

$$v_1(t_1) = v - (a\gamma/2c^2)(2c + v)d + ( ) d^2 + \dots \quad (102)$$

The resulting expansion of  $\mathbf{E}_1(t)$  is

$$\mathbf{E}_1(t) = (e/2d^2) [1 - (a\gamma^3/c^2)d + ( ) d^2 + \dots] \hat{z}, \quad (103)$$

while the corresponding calculation for  $\mathbf{E}_2(t)$  yields

$$\mathbf{E}_2(t) = - (e/2d^2) [1 + (a\gamma^3/c^2)d + ( ) d^2 + \dots] \hat{z}. \quad (104)$$

Finally, the self-force on the dumbbell is

$$\mathbf{F}_{\text{self}} = (e/2)(\mathbf{E}_1 + \mathbf{E}_2) = [ - \gamma^3(e^2/2dc^2)a + \dots ] \hat{z}, \quad (105)$$

and the self-force-derived mass

$$m_s = \gamma^3(e^2/2dc^2) \quad (106)$$

now includes the appropriate factor of  $\gamma^3$ .<sup>22</sup>

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<sup>1</sup>A fourth technique is suggested by the equivalence principle. See T. H. Boyer, *Am. J. Phys.* **47**, 129 (1979).

<sup>2</sup>For motion in a straight line, the relativistic formula  $F = dp/dt$  yields  $F = d/dt(\gamma m v) = d\gamma/dt m v + \gamma m dv/dt = \gamma^3 m a$ .

<sup>3</sup>This expectation is skillfully exploited in T. H. Boyer, *Am. J. Phys.* **46**, 383 (1978). However, in the light of the present paper Boyer was fortunate to have chosen transverse motion.

<sup>4</sup>F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Reading, MA, 1965), Chap. 2; A. Pais, "The Early History of the Theory of the Electron: 1897-1947," in *Aspects of Quantum Theory*, edited by A. Salam and E. Wigner (Cambridge U. P., London, 1972).

<sup>5</sup>H. Poincaré, *Co. R. Acad. Sci. Paris* **140**, 1504 (1905); *Rend. Circ. Mater.* (Palermo) **21**, 129 (1906).

<sup>6</sup>See F. Rohrlich, Ref. 4, p. 17.

<sup>7</sup>F. Rohrlich, *Am. J. Phys.* **28**, 639 (1960); **38**, 1310 (1970).

<sup>8</sup>J. W. Zink, *Am. J. Phys.* **34**, 211 (1966); A. Gamba, *ibid.* **35**, 83 (1967).

<sup>9</sup>D. J. Griffiths and E. W. Szeto, *Am. J. Phys.* **46**, 244 (1978). See also Boyer (Ref. 3) and Pearle (Ref. 10).

<sup>10</sup>Two outstanding articles on classical electron theory have recently appeared in *Electromagnetism: Paths to Research*, edited by D. Teplitz (Plenum, New York, 1982). Chapter 6 is S. Coleman's widely known but hitherto unpublished monograph *Classical Electron Theory from a Modern Standpoint*, and Chap. 7 (*Classical Electron Models*) is a review by P. Pearle. The latter includes an extensive bibliography.

<sup>11</sup>E. M. Purcell, *Electricity and Magnetism* (McGraw-Hill, New York, 1963), p. 51.

<sup>12</sup>J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975), p. 239.

<sup>13</sup>E. M. Purcell, Ref. 11, p. 214.

<sup>14</sup>E. M. Purcell, Ref. 11, p. 160.

<sup>15</sup>D. J. Griffiths, *Introduction to Electrodynamics* (Prentice-Hall, Englewood Cliffs, NJ, 1981), p. 372.

<sup>16</sup>The  $d^3$  term in the expansion for  $\mathbf{F}_{\text{self}}$  gives the radiation reaction force (Griffiths and Szeto, Ref. 9).

<sup>17</sup>This case was analyzed by G. S. Cardell, thesis, Reed College, 1981 (unpublished). See also Boyer (Ref. 3).

<sup>18</sup>This argument and the idea for the Appendix are due to D. G. Hoffman.

<sup>19</sup>E. M. Purcell, Ref. 11, p. 51.

<sup>20</sup>This is the famous factor of 4/3 discussed in Refs. 4 and 10.

<sup>21</sup>Note added in proof: Pearle (Ref. 10) has recently shown that the discrepancy is resolved in the case of the spherical shell by inclusion of Lorentz contraction in the self-force calculation.

<sup>22</sup>Most of the material in this paper appears in more detailed form in R. E. Owen, thesis, Reed College, 1982 (unpublished).