The Impossibility of Vagueness

______I wish to present a proof that vagueness is impossible. Of course, vagueness is possible; and so there must be something wrong with the proof. But it is far from clear what is wrong and, indeed, all of the assumptions upon which the proof depends are ones that have often been accepted. This suggests that we may have to radically alter our current conception of vagueness if we are to make proper sense of what it is.

The present investigation was largely motivated by an interest in what one might call the 'global' aspect of vagueness. We may distinguish between the indeterminacy of a predicate in its application to a single case (the local aspect) and in its application to a range of cases (the global aspect); and the question arises as to whether one might understand the indeterminacy of a predicate in its application to a range of cases in terms of its indeterminacy in application to a single case. Judged from this perspective, the result can be seen to show that there is no reasonable way in which this might be done.

But the result can also be seen to arise from an interest in higher-order vagueness. It has often been observed that there is a difficulty in conceiving of indeterminacy in the presence of higher-order vagueness. For it cannot consist in these cases being borderline and those other cases not being borderline, for we would then have a sharp line between the borderline cases and the non-borderline cases, contrary to the existence of higher-order vagueness; and, for similar reasons, it cannot be taken to consist in these cases being borderline borderline and those other cases not being borderline borderline or in something else of this sort. But it is hard to pin this difficulty down and, certainly, the failure of one particular attempt to characterize indeterminacy in the presence of higher order vagueness does nothing to establish a failure in principle. Judged from this perspective, the present result can be seen to provide a vindication of those (such as Graff-Fara [2003], Sainsbury [1991], Wright ([1987], [1992])) who have thought that the existence of higher order vagueness does indeed stand in the way of having a reasonable conception of what indeterminacy might be.

I begin the paper by giving an informal presentation of the result and I then consider the various responses that might be made to it. Most of these are found wanting; and my own view, which I hint at rather than argue for, is that is only by giving up on the notion of single-case indeterminacy and by modifying the principles of classical logic that the impossibility result can properly be evaded. There are two appendices, one providing a formal presentation and proof of the impossibility theorem and the other giving a counter-example to the theorem under a certain relaxation of its assumptions. The mathematics is not difficult but those solely interested in the philosophical implications of the results should be able to get by without it.

I have made no attempt to compare the present proof of impossibility to the arguments in Graff-Fara [2003], Sainsbury [1991], and Wright [1987]. Both approaches are based upon the general idea that there is some kind of incompatibility between a claim of indeterminacy and the existence of a sharp boundary. But my own approach leaves open how a claim of indeterminacy might be formulated and it rests upon a more general conception of what might constitute a sharp boundary. In particular, I do not assume that the members of a sorites are ordered in a series or that the 'adjacent' members are somehow related to one another.

§1 Informal Presentation of the Theorem

______Suppose we are presented with a sorites series - a series of bald men b_0, b_1, ...,b_n, say, which ranges through gradual increments from the first b_0, who is hairless, to the last b_n, who is very hairy. Let p_0, p_1, ..., p_n be the corresponding propositions that b_0 is bald, that b_1 is bald, ..., and that b_n is bald. There are then three things that we would be correct - and, indeed, utterly confident - in asserting. One is p_0, that the first man b_0 is bald, another is not-p_n, that the last man p_n is not bald, and the third is that the predicate 'bald' is not completely determinate in its application to the men b_0, b_1, ...,b_n of the series.

Call the last of these claims 'the indeterminacy claim'. It is not altogether clear what is involved in making such a claim. But it does seem clear that its assertion should not be compatible with a joint resolution of each case. Suppose that one is presented with a 'forced march' - one is successively asked 'Is b_0 bald?', 'Is b_1 bald?', ..., 'Is b_n bald?'; and suppose that, upon being presented with a forced march, one gives either the positive answer 'Yes' or the negative answer 'No' to each of the questions. Where there are 25 men, for example, one might respond 'Yes' to the first 12 questions and 'No' to the remaining 13 or perhaps 'Yes' to the first 13 questions and 'No' to the remaining 12. In such a case, it would surely be incompatible with these answers to go on to assert that the predicate 'bald' was indeterminate in its application to the various men. Some kind of contradiction would be involved in providing a positive or negative answer to each of the questions and yet claiming the predicate to be of indeterminate application.

It will not be important, in what follows, to insist that the relevant notion of incompatibility should be *logical* incompatibility. Given an indeterminacy claim, there may perhaps be no logical incompatibility with a joint resolution of each case. But it does seem very plausible that there will be an incompatibility in a broadly conceptual sense - that the assertion of indeterminacy, by virtue of its very content and perhaps also by virtue of its being an assertion of that content, will exclude a joint resolution of the cases.

The notion of incompatibility, whether logical or not, is naturally taken to be aligned to a corresponding notion of consequence or commitment by means of the following principle:

(*) the assertion of various propositions are jointly incompatible iff their assertion commits one to a contradiction, i.e. to a given proposition and its negation.

Thus the assertion of the indeterminacy claim will be incompatible with the complete resolution of each case since the claim that there is a complete resolution of each case will commit one to the predicate 'bald' *not* being indeterminate in its application to the various cases, in contradiction to the indeterminacy claim. Although this principle is very plausible, it is not, in fact, essential to our argument - which could be stated more directly in terms of consistency, without regard to its connection with consequence. But the other principles involved in the argument are commonly framed in terms of consequence and, in order to preserve the sense of familiarity, I have thought it desirable also to frame my argument in these terms.

The notion of consequence or commitment is normally taken to conform to the principle of reductio ad absurdum:

(**) if the assertion of the propositions P_1, P_2, ... along with the proposition Q commit one to a contradiction, then the assertion of the propositions P_1, P_2, ... alone commit one to not-Q.

But as has often been observed, this principle is not at all plausible in the context of vagueness. For suppose that one is willing to talk, in such a context, of a proposition's being definitely the case or of its not being definitely the case. The assertion that a given proposition is not definitely the case is then presumably incompatible with asserting that it is the case, i.e. their joint assertion will commit one to the contradiction that it is definitely the case and not definitely the case. So by reductio, the assertion that the proposition is not definitely the case will commit one to the conclusion that it is not the case and, since the assertion that it is not the case will commit one to its not being definitely the case, the distinction between the proposition's not being the case and its not being definitely the case will disappear.

One might plausibly, and familiarly, explain these counter-examples to reductio along the following lines. In asserting some propositions P_1, P_2, ..., one is committed to more than their actual content, one is also committed to their being definitely the case, definitely definitely the case, definitely definitely definitely the case, and so on. We might say that a proposition is *super-definitely* the case if it is the case, definitely the case, definitely definitely the case, and so on ad infinitum. Then in asserting some propositions, one is also committed to their being super-definitely the case and it is because of this further content that one cannot infer from the inconsistency of Q with some other propositions that Q alone (apart from its further content) is not the case. In the case of the assertion of not definitely P and P, for example, what explains their joint incompatibility is the straightforward conflict between P's being definitely the case and its not being definitely the case; and no inference from not definitely P to not-P is therefore justified.

This suggests that there is an underlying notion of consequence which *will* conform to reductio and for which the commitment of P_1, P_2, ... to Q will amount to the Q along with its supplementary content Q* being a consequence of P_1, P_2, ... along with their supplementary content P_1*, P_2*, ... In the case of vagueness, there is no reason to think that any further content beyond the super-definiteness of the propositions in question will be relevant to whether there is a commitment. We are therefore led to the following principle:

(***) the assertion of P_1, P_2, ... commits one to Q iff Q's being super-definite is a consequence of its being super-definite that P_1, super-definite that P_2, ...[1]

I have used the terms 'commitment' and 'consequence' to mark the distinction between the two kinds of entailment (with the term 'commitment' suggesting that there might be a connection with the pragmatics of assertion). There is a related distinction between 'compatibility' and 'consistency', where the compatibility of certain propositions is their failure to commit one to a contradiction (as in (*) above) and the consistency of certain propositions is their failure to have a contradiction as a consequence. Certain propositions will then be compatible just in case their being super-definitely the case is consistent.

We must now return to the question of how one might respond to a forced march. We have already remarked that asserting the indeterminacy of the predicate 'bald' in application to the men of the sorites series should not be compatible with a positive or negative answer to each question within a forced march. But something more general would also appear to hold. For suppose one were to respond to a forced march by saying that each of the first nine men, say, were definitely bald, that each of the next three men were borderline bald, i.e. neither definitely bald nor definitely not bald, and that

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[1]The argument given against such reductions towards the end of Graff-Fara [2003] is not relevant to the present case since she assumes, somewhat strangely, that the supplementary content will always be the same regardless of which propositions P_1, P_2, ... are in question. Indeed, her semantic characterization of the consequence relation for the supervaluationist directly supports the kind of reduction I have in mind.

each of the remaining men were definitely not bald. Then this would presumably also be incompatible with an indeterminacy claim. For a sharp line is still being drawn, not now between the men who are bald and the men who are not bald, but between the men who are definitely bald and the men who are borderline bald and, in addition, between the men who are borderline bald and the men who are definitely not bald. And the existence of sharp lines at this 'higher' level would appear to be as much in conflict with a claim of indeterminacy, as it is most naturally understood, as the existence of a sharp line at the 'lower' level.

What goes for sharp lines at this higher level would appear to extend to sharp lines at higher levels still. It would not do, for example, to respond to each question within a forced march either with the response that the man is definitely definitely bald or with the response that he is not definitely definitely bald and not definitely definitely not bald or with the response that he is definitely definitely not bald (cf. Sainsbury [1991], 168-900).

The more general point would appear to be this. Consider any series of responses to a forced march - such as 'Yes, ..., Yes, No, ..., No' or 'Definitely Yes, ..., Definitely Yes, Borderline, ..., Borderline, Definitely No, ..., Definitely No'. Call such a series of responses *sharp* if (a) not all of the responses are the same and (b) any two responses that are not the same are inconsistent with one another. Then a claim of indeterminacy should exclude a sharp response to a forced march; it should not be possible to making the indeterminacy claim compatibly with giving a sharp response.

We have so far formulated what are, in effect, two requirements on a satisfactory statement of indeterminacy. The first of these, which we may call the *Incompatibility Requirement*, is that the indeterminacy claim should be incompatible, in the intended sense, with a sharp response to the forced march. The second, which we may call the *Compatibility Requirement*, is that the indeterminacy claim should be compatible with a positive response to the first question within a forced march and a negative response to the last. For, as we have observed, it will be correct to make an indeterminacy claim in regard to a sorites series and also correct to give a positive response to the first case and a negative responses to the last case; and, if it is *correct* to make the claim and to give these responses, then it will be certainly be *compatible* with making the claim that one give these responses.

We can now state the impossibility result:

<u>There is no putative claim of indeterminacy that satisfies the compatibility and incompatibility requirements, i.e., there are no propositions that are both compatible with a positive and a negative response to the extreme cases of a sorites series and yet incompatible with any sharp response.</u> Vagueness will therefore be impossible in so far as there is nothing that can meet the demands upon which its existence would appear to depend.

A precise formulation and proof of this result is given in the appendix. Not all of the details of the formulation and the proof will be important to the subsequent philosophical discussion but some of them should be noted.

We should note, in the first place, that the proof of the result rests upon two principal assumptions. The first of these is that the notions of commitment and consequence should conform to (***) above, i.e. that P_1, P_2, ... commit one to Q iff Q being super-definite is a consequence of P_1, P_2, ... being super-definite. The second is that the notion of consequence should conform to (**) above, i.e. that not-Q should be a consequence of P_1, P_2, ... if P_1, P_2, ... and Q are inconsistent.

There are some ancillary assumptions upon which the proof depends but which are much less open to doubt. They are that:

(I) Consequence is subject to the usual structural rules (such as that P is a consequence of P and that R is a consequence of Q, P_1, P_2, ... if it is a consequence of P_1, P_2, ... alone);

(II) Conjunction is subject to the usual introduction and elimination rules;

(III) The definitely operator is subject to the principles of the modal logic T (what is definitely the case is the case and if Q is a consequence of P_1, P_2, ... then Definitely Q is a consequence of Definitely(P_1), Definitely(P_2), ...).

Given the supposition that an alleged indeterminacy claim is compatible with a positive and negative response to the extreme cases of the sorites series, the proof actually constructs a sharp response that is compatible with those propositions; and this sharp response takes a particular form. For in the case of each man in the series, we may say either that he is super-definitely bald or that he is super-definitely not bald or that he is super-indefinitely bald, i.e. neither super-definitely bald nor super-definitely not bald. Thus it is a sharp response of this very particular form that we can show is not excluded.

§2 Challenging the Assumptions

The impossibility result purports to show that vagueness is impossible. Since vagueness clearly is possible, there must be something wrong with one or more of the assumptions upon which the proof of the result depends. But which? And how?

It should be noted that the theorem is not directed towards any particular theory of vagueness and so it is incumbent on *anyone* with a view on vagueness to show how their view is to be reconciled with the result. We should also note that, even if one were to reject an assumption upon which the proof of the result depends, one would still face the problem of saying how the claim of indeterminacy is to be stated. The result points to a genuine difficulty in formulating inderminacy claims; and when one examines the usual way of formulating these claims, they can immediately be seen to be wanting. It is common, for example, to take a predicate to be indeterminate in its application to a range of objects just in case one of those objects is a borderline case of the predicate. But the predicate's admitting a borderline case is compatible with the sharp response in which one case is borderline and the rest are not; and similarly for other formulations of this sort. Thus even if the impossibility result could somehow be finessed, it would still be far from clear how a claim of indeterminacy is actually to be formulated.

In considering the various responses to the impossibility result, let me begin with some that may naturally occur to the reader but which do not strike me as so promising and then go on to consider some others which need to be taken more seriously.

One possible misgiving is over the question of whether a satisfactory formulation of the indeterminacy claim should conform to the compatibility and incompatibility requirements. Now the compatibility requirement would appear to be non-negotiable but the need for the incompatibility requirement is not so clear. For one may well think that a claim of indeterminacy will be compatible with a sharp division of the cases into those who are definitely bald, those who are borderline, and those who are definitely not bald. After all, would not the existence of borderline case guarantee that the predicate was indeterminate, regardless of what else might obtain?

It might be doubted whether in a case of an actual vague predicate such as 'bald' it would ever be correct to make a sharp division of this sort. For one might well think that whatever it was that prevented one from correctly making a bipartite division of the cases into 'bald' and 'not bald' would also prevent one from correctly making a tripartite division of the cases into 'definitely bald', 'definitely not bald' and 'borderline'. But let it be allowed that a case of this sort might arise and that there was even a sense of 'indeterminate' in which it might then be correct to say that the predicate was indeterminate. Still, one would have to admit that there was a more radical, and more usual, way in which a predicate was capable of being indeterminate and which was incompatible with any kind of sharp response, be it of an ordinary or of a less ordinary sort; and it is this more radical way of being indeterminate that is here in question (cf. Sainsbury [1991], 179).

Another possible misgiving concerns our account of compatibility. It might be thought that we are not justified in distinguishing in the way we have between consistency and compatibility and that the notion of consistency should be used throughout in stating the compatibility and incompatibility requirements. But this does not help at all, because if we take 'definitely P' simply to mean 'P', compatibility will collapse into consistency and the result will still apply. Using a notion of compatibility in contrast to the notion of consistency makes it in fact harder rather than easier to establish the impossibility.

It might, on the other hand, be thought that our account of compatibility needs to be strengthened rather than weakened. But how? One possibility is to insist that the propositions whose compatibility is being considered should not only be super-definite but also definitely super-definite, definitely definitely super-definite, super-super-definite, and so on. But given the standard logic of the definitely operator, these supposed strengthenings are not strengthenings at all. For consider the claim that P is super-definite. This amounts to the claim: (P & definitely P & definitely definitely P & ...). Such a claim commits one to (definitely P & definitely definitely P & ...), of course. Moreover, it is generally true that (definitely P_1 & definitely P_2 & ...) commits one to definitely (P_1 & P_2 & ...); and so, in particular, (definitely P & definitely definitely P & ...) will commit one to definitely(P & definitely P & ...), which is just to say that it is definitely super-definite that P. Thus nothing is to be gained by piling on further applications of the definitely operator.

One might, of course, insist that the propositions should also be hyper-definite, where this is something stronger than being super-definite, or that it should be hyper-definite to the 'second order', where this is something stronger still than being hyper-definite or hyper-hyper-definite or hyper-definite to any finite degree n. If all of the orders of definiteness can be wrapped up into a single 'transcendental' order of definiteness, then the result can still be reinstated using the transcendental order of definiteness in place of the original low-level notion. But one might have the view that, even though one can make a compendious assertion of definiteness by way of a 'scheme' that runs through all the possible ways to understand the notion, the content of this compendious assertion is not itself capable of being denied (similar positions have been adopted in connection with the semantic and set-theoretic paradoxes). The proof of the result cannot then be reinstated since it requires the application of reductio to the content of the compendious assertion, which must therefore be something that can be denied.

Intriguing as this suggestion may be, I doubt that it can be used to get round the essential difficulty. For it is plausible that the claim of indeterminacy will be formulated using notions of hyper-definiteness up to a certain order (and this would appear to be especially true if it is the kind of claim that can be denied and is not merely a 'scheme'). But higher orders of hyper-definiteness will then be irrelevant to the existence of an incompatibility between the indeterminacy claim and a sharp response. For the result will tell us that there is a sharp response compatible with the indeterminacy claim as long as the iterations of the hyper-definitely operator are confined to the given order; and this compatibility will remain even when arbitrarily high order iterations of the hyper-definitely operator are allowed.[2]

Might there be other strengthenings that do not make use of a definitely operator? Possibly, but it is hard to see what they might be if they are to be peculiarly relevant to the use of a vague language and to a possible conflict with the indeterminacy claim.

A third, somewhat more technical, misgiving concerns the 'structural' rules that we have presupposed in talking of consequence. For the proof requires in effect that consequence conforms to the usual structural rules:

Identity Any proposition is a consequence of itself;

Weakening Consequence hold under the addition of superfluous premisses;

Permutation The order of the premisses is irrelevant;

Contraction The repetition of premisses is irrelevant

Cut Consequences can be chained with the conclusion of one relationship of consequence serving as the premiss of another.[3]

But various philosophers in the tradition of relevance logic have objected to some of these rules.

The force of these objections can to a large extent be deflected by adopting a conjunctive interpretation of consequence. One takes Q to be a consequence of the propositions P_1, P_2, ... in the conjunctive sense just in case Q is a consequence, in the given sense, of their conjunction P_1 & P_2 & .... The rules of Weakening, Permutation and Contraction are all then very plausible. The objection, to the extent that it still arises, will then concern the rule of reductio. Thus it may be granted that P and not-P are consequences of (P & not-P & Q) but not thereby granted that not-Q is a consequence of (P & not-P). I do not know if there is a version of the result that can be proved for the various forms of relevant consequence, but my suspicion is that the usual approaches to relevance logic will be of no help in allowing us to evade the result.

A final misgiving is that we seem to have presupposed that the indeterminacy claim can be formulated exclusively in terms of the definitely operator (and the truth-functional connectives). But is it not possible that it can formulated in other terms or that the indeterminacy of a predicate over a range of objects might even be adopted as primitive? Now although the formal presentation of the theorem might appear to presuppose that the indeterminacy claim is to be formulated exclusively in terms of the definitely operator, this is not in fact so. The result is indifferent to how indeterminacy might be formulated and will hold under any extension of the language, and even under the addition of a global indeterminacy operator, as long as the required relationships between consequence, definiteness and the

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[2]The technical reason is this. Write the hyper-definitely operator of order α as $D\langle\alpha\rangle$ and, where Δ_0 is the indeterminacy claim, let α_0 be an upper bound for all the α for which $D\langle\alpha\rangle$ occurs in Δ_0. The proof produces a sharp response R (also bound by α_0) which is compatible with Δ_0 under arbitrary iterations of $D\langle\alpha\rangle$ for $\alpha \leq \alpha_0$. But if R were to be incompatible with Δ_0 under arbitrary iterations of the $D\langle\alpha\rangle$, for *any* α, then it should remain incompatible under the 'identification' of $D\langle\alpha\rangle$ with $D\langle\alpha_0\rangle$ for all $\alpha > \alpha_0$.

[3]I say 'in effect' since Permutation and Contraction are not stated as such but are built into the assumption that consequence is a relation that holds between a *set* of premisses and a given conclusion.

truth-functional connectives are all preserved.

______I turn now to the more serious responses. There are perhaps three in all. The first again questions our account of compatibility. For it might be thought that we go astray in insisting that the compatibility of various propositions should require the consistency of their being super-definite. All that is strictly required is the consistency of their being definite, but not necessarily the consistency of their being *super*-definite or even the consistency of their being *definitely* definite. Call this non-iterative form of compatibility *weak* as opposed to *strong*.[4]

It might now be conjectured that the compatibility and incompatibility requirements can both be met, as long as the compatibility in question is weak rather than strong. Somewhat surprisingly, it turns out that this is so (a specific way of formulating a relevant indeterminacy claim and a proof that it satisfies the requirements is given in the second part of the appendix).

Do we therefore have a satisfactory response to the impossibility result? The critical question is whether, in asserting a given proposition, we are committed to its being definite. For if we are, then we will be committed to its being definitely definite, definitely definitely definite etc., and thereby committed to its being super-definite. Now it may be conceded that there are ways of understanding the definitely operator for which the rule of D-Introduction is not at all obvious and may not even be correct. On one natural understanding of 'definitely', for example, to say that definitely P is to say that there is no doubt that P. And although it may be thought that one is only correct in asserting P if there is in fact no doubt that P, it does not follow that in asserting P one should thereby be willing to assert that thereis no doubt that P, no doubt that there is no doubt, and so on.

However, the fact that there is an interpretation of 'definitely' for which D-Introduction fails does little to show that there is no interpretation for which it holds; and, indeed, it seems to me that the most natural interpretation of 'definitely' in the context of vagueness is one for which the rule will in fact hold. For the most natural interpretation is one in which it is cognate with the notion of being a borderline case, where to say that x is definitely F in the relevant sense is to say that it is F and not a borderline case of F (or where equivalently, under very plausible assumptions, to say that x is a borderline case of F is to say that it is not definite that it is F and not definite that it is not F). But now the assertion that a man is bald, let us say, will surely commit one to his not being a borderline case of a bald man. For how could one sensibly assert that a given man is bald and yet not thereby be willing to deny that he is a borderline case of a bald man? But given that this is so, it will then follow directly from the equivalence above that the man is definitely bald; and the rule of D-Introduction will be vindicated.

Indeed, it seems to me that the interpretations of the definitely operator for which D-Introduction fails will be irrelevant to the intended interpretation of the result. For the plausibility of the incompatibility requirement rests upon supposing that the individual responses that constitute a sharp response should be *genuine* responses to the questions that arise within a forced march. Suppose that a party to a forced march in the predicate 'bald' were to respond in each case by saying 'I do not doubt that the person is bald' or 'I do doubt that the person is bald'. Such a putative response is sharp and yet surely compatible with acknowledging the indeterminacy of the predicate. But this is because the responses in question are not genuine responses; they do not bear exclusively upon the question of whether the given person is bald. If someone were to respond by declaring that he doubts that the

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[4]Williamson ([1994], chapter 8) holds a view of this sort.

person is bald, then one can legitimately go on to ask 'That may be so, but what of my question - *is he bald*?'. It is the use of 'definitely' which might figure in a genuine response to a forced march that will be relevant to the interpretation of the result; and such a use will be subject to D-Introduction. For once one has declared that a given person is bald, there is nothing more to be said on the question of whether he is bald; and so the claim that he is definitely bald, if it is to have any bearing upon the question, must already be implicit in what one has said.

Another, much more radical, response is to give up on the idea of single-case indeterminacy. We have in effect presupposed that there is a notion of a proposition's being definitely the case which conforms to the usual modal principles and whose use might help constitute a genuine response to the question of whether a given individual has a certain property F. But it might be doubted whether there is such a notion. Even if it is granted that there is a use of 'definitely' under which it might be correct to say that a given person is not definitely bald and not definitely bald, this will not be a use, according to the present proposal, that genuinely bears upon the question of whether he is bald and that might therefore constitute part of a sharp response with which a claim of indeterminacy should be incompatible.

I am not unsympathetic to this proposal but it seems to me that its adoption requires that we should also give up reductio (for the notion of consequence). For in the absence of definiteness, we no longer have any basis for distinguishing between consistency and compatibility. What this means, in effect, is that we may connect compatibility with consistency and definiteness in the way we have as long as we identify 'definitely P' with 'P'. But, as we have observed, the impossibility result will still obtain under this identification. And so how is it to be avoided except by giving up reductio?

The third response is to give up reductio. Again, I am not unsympathetic to this proposal but it seems to me that its adoption requires that we also give up the idea of single-case indeterminacy (so that giving up either one requires giving up the other). For suppose that a contradiction is a consequence of the proposition P along with some other assumptions. Then why should we not take not-P to be a consequence of the other assumptions? Given a concept of truth under which we might distinguish between a proposition's not being true and its being false, then the most plausible answer is that the contradiction is a consequence in effect of the *truth* of P, so that all that can properly be inferred by reductio from the other assumptions is that P is not true - not that P is false. But if this is the explanation, then we can identify a proposition's being definitely the case with its being true and reinstate the impossibility result using this weakened form of reductio (where what is inferred is not not-P but not-definitely-P).

We therefore end up with the remarkable conclusion that the impossibility is to be avoided by both giving up the idea of single-case indeterminacy and rejecting the principle of reductio. It has often been supposed that the presence of vagueness might require some departure from classical logic. But it has been almost universally assumed that the characterization of vagueness will require an appeal to the notion of a borderline case. If I am right, the acknowledgment of such a notion, rather than helping us to understand vagueness, stands in the way of even seeing how it could exist.

Appendix

We first provide a formal statement and proof of the impossibility theorem. We presuppose, by way of background, an infinitary sentential language $\mathscr{L}^{\infty}$(D). The *symbols of* $\mathscr{L}^{\infty}$(D) are:

(i) the sentence-letters $p_1, p_2, \ldots$;
(ii) the negation operator ¬;
(iii) the conjunctive operator ⋀; and
(iv) the definitely operator □.

The *formulas of* $\mathscr{L}^\infty$(D) are defined according to the following rules:

(i) each sentence letter is a formula;
(ii) if A is a formula, then so is ¬A;
(iii) if Δ is a countable (i.e. either a finite or countably infinite) set of formulas, then ⋀Δ is a formula;
(iv) if A is formula, then so is DA.

It is largely for convenience that we take the conjunction operator ⋀ to apply to a *set* rather than a *sequence* of formulas. Formally, we might think of ⋀Δ as the ordered pair <⋀, Δ> but, when Δ = $\{A_1, A_2 \ldots\}$, we may write ⋀Δ, more intuitively, as $A_1 \wedge A_2 \wedge \ldots$.

We use the following abbreviations:

(B ∧ C) for ⋀{B, C};
IA for ¬DA ∧ ¬D¬A.
D^0A for A, and $D^{n+1}A$ for DD^nA, n = 0, 1, ...;
$D^\infty A$ for $\bigwedge\{D^n A: n = 0, 1, 2, \ldots\}$; and
$I^\infty A$ for $\neg D^\infty A \wedge \neg D^\infty \neg A$.

Thus D^nA is the formula:

$$\overbrace{D \ldots D}^{n}A,$$

while $D^\infty A$ is the formula:

$$A \wedge D^1A \wedge D^2A \wedge \ldots.$$

Given a set Δ, we let DΔ be {DA: A ∈ Δ}. We say that the set of formulas Δ is D-*closed* if DΔ ⊆ Δ, i.e. if DA ∈ Δ whenever A ∈ Δ; and we let Δ^D be the smallest D-closed set to contain Δ. We sometimes use A^D for $\{A\}^D$, i.e. for $\{D^n A: n = 0, 1, 2, \ldots\}$. Evidently, $\Delta^D = \bigcup\{A^D: A \in \Delta\} = \{D^n A:$ for $A \in \Delta$ and $n = 0, 1, 2, \ldots\}$ and $(\Delta^D)^D = \Delta^D$.

We take the formulas of language $\mathscr{L}^\infty$(D) to be governed by a relation of consequence |- that holds between a countable set of formulas Δ and a single formula A (though there would be no problem in extending the relation to uncountable sets of formulas). We say Δ |- Γ if Δ |- A for each A in Γ and we use lists of formulas and sets to the right or left of |- in an obvious way.

The consequence relation is governed by the following three groups of rules:

(I) Structural Rules

Identity A |- A;
Weakening if Δ |- A and Δ ⊆ Γ then Γ |- A;
Cut If Δ |- Γ and Γ, Θ |- A then Δ, Θ |- A.

(II) Rules for ¬ and ⋀

¬-Introduction (Reductio) if Δ, A |- B, ¬B then Δ |- ¬A.

<u>$\bigwedge$-Introduction</u> if Δ |- Γ then Δ |- $\bigwedge\Gamma$

<u>$\bigwedge$-Elimination</u> $\bigwedge\Delta$ |- A if $A \in \Delta$.

<u>Rules for D</u>

<u>D-Elimination</u> DA |- A

<u>D-Distribution</u> if Δ |- A then DΔ |- DA.

Note that Δ |- Δ by Identity and Cut and so Δ |- $\bigwedge\Delta$ by $\bigwedge$-Introduction. Use of the structural rules will often be implicit.

We define *definite consequence* or *commitment* by:

Δ $|\!-^D$ A if Δ^D |- A^D.

Definite consequence only requires strengthening on the left:

<u>Lemma 1</u> If Δ $|\!-^D$ A iff Δ^D |- A.

<u>Proof</u> The left-to-right direction is trivial. For the right-to-left direction, assume Δ^D |- A. By D-Distribution, D(Δ^D) |- DA. But $D(\Delta^D) \subseteq \Delta^D$. So by Weakening, Δ^D |- DA. Iterating the argument, Δ^D |- D^nA for n = 2, 3, ... But then Δ^D |- D^nA for n = 0, 1, 2, ...; and so Δ^D |- $D^\infty A$ by $\bigwedge$-Introduction.

<u>Lemma 2</u> $|\!-^D$ conforms to the structural rules of Identity, Weakening and Cut .

<u>Proof</u> We consider each rule in turn.

<u>Identity</u> A^D |- A by Identity and Weakening for |-; and so A $|\!-^D$ A, by lemma 1.

<u>Weakening</u> Suppose Δ $|\!-^D$ A and $\Delta \subseteq \Gamma$. Then $\Delta^D \subseteq \Gamma^D$. So by Weakening for |-, Γ^D |- A; and hence Γ $|\!-^D$ A (again by lemma1).

<u>Cut</u> Suppose Δ $|\!-^D$ Γ and Γ, Θ $|\!-^D$ A. Then Δ^D |- Γ^D and Γ^D, Θ^D |- A^D. By Cut for |-, Δ^D, Θ^D |- A^D; and consequently, Δ, Θ $|\!-^D$ A.

It is evident that definite consequence conforms to the rule of D-Introduction, i.e. that A $|\!-^D$ DA, since A^D |- DA by Identity and Weakening. I note without proof that $|\!-^D$ is the smallest relation to contain |- and to conform to the structural rules and D-introduction.

Reductio holds in the following modified form for definite consequence:

<u>Lemma 3</u> If Δ, A $|\!-^D$ B, $\neg$B then Δ $|\!-^D$ $\neg D^\infty A$.

<u>Proof</u> Suppose Δ, A $|\!-^D$ B, $\neg$B. Then Δ^D, A^D |- B, $\neg$B. By $\bigwedge$-Elimination, $D^\infty A$ |- A^D and so, by Cut, Δ^D, $D^\infty A$ |- B, $\neg$B. But then by reductio for |-, Δ^D |- $\neg D^\infty A$; and consequently, Δ $|\!-^+$ $\neg D^\infty A$.

To state the impossibility result, we need some further terminology. A set of formulas Δ is said to be *inconsistent* if, for some formula B, Δ |- B and Δ |- $\neg$B and Δ is otherwise said to be *consistent*. Likewise, Δ is said to be *incompatible* if, for some formula B, Δ $|\!-^+$ B and Δ $|\!-^+$ $\neg$B and Δ is otherwise said to be *compatible*. Δ is said to be *inconsistent* (or *incompatible*) *with* the set Γ if $\Delta \cup \Gamma$ is inconsistent (or incompatible).

Let p be an any sentence-letter, fixed for the purposes of the following discussion. Then an *individual response* is a formula A(p) whose sole sentence-letter is p; and the formula A is a *response to* the question of B if it is of the form A(B), where A(p) is an individual response. A *collective response* is a sequence A(p), A(p), ..., A(p) of individual responses; and A_1, A_2, ..., A_n is said to be a *collective response to* B_1, B_2, ..., B_n if A_1, A_2, ..., A_n are respectively of the form $A_1(B_1)$, $A_2(B_2)$,

..., $A_n(B_n)$, where $A_1(p)$, $A_2(p)$, ..., $A_n(p)$ is a collective response.

We say that the collective response $A_1, A_2, ..., A_n$ is *sharp* if:

(i) $A_i \neq A_j$ for some $i, j \leq n$;

(ii) A_i is inconsistent with A_j or $A_i = A_j$ for $1 \leq i < j \leq n$.

In a sharp response, we either give the same answer to each question or inconsistent answers, with at least two of the answers not being the same. We may similarly talk of a *sharp* response *to* $B_1, B_2, ..., B_n$.

We are now in a position to state the result:

Theorem 1 Take any formulas $B_0, B_1, ..., B_n$, $n > 0$. Then there is no set of formulas Δ_0 which is compatible with B_0 and $\neg B_n$ and yet incompatible with any sharp response to $B_0, B_1, ..., B_n$.

Proof The proof is somewhat reminiscent of the proof of Lindenbaum's Lemma. Take any formulas B_0, $B_1, ..., B_n$ and any set of formulas Δ_0 compatible with B_0 and $\neg B_n$. We show that Δ_0 is compatible with a sharp response to $B_0, B_1, ..., B_n$.

To this end, we 'blow up' Δ_0 to a set Δ_n from which a compatible sharp response can be more readily discerned. We let $\Delta_1 = \Delta_0 \cup \{B_0, \neg B_n\}$ and, for $k = 1, 2, ..., n-1$, we let:

$\Delta_{k+1} = \Delta_k \cup \{B_k\}$ if Δ_k is compatible with B_k;
$= \Delta_k \cup \{\neg B_k\}$ if Δ_k is compatible with $\neg B_k$
$= \Delta_{k-1}$ otherwise.

It is evident from the construction that:

(1) Δ_k is compatible for $k = 0, 1, ..., n$; and

(2) $\Delta_k \subseteq \Delta_l$ for $0 \leq k < l \leq n$.

Using Δ_n, we define a collective response $A_0(p), A_1(p), ..., A_n(p)$ (and a corresponding collective response $A_0(B_1), A_1(B_2), ..., A_n(B_n)$ to $B_0, B_1, ..., B_n$). Where $k = 0, 1, ..., n$:

(a) $A_k(p) = D^\infty(p)$ if $B_k \in \Delta_n$,

(b) $A_k(p) = D^\infty(\neg p)$ if $\neg B_k \in \Delta_n$, and

(c) $A_k(p) = I^\infty(p)$ otherwise.

The collective response is well-defined since if $B_k \in \Delta_n$ and $\neg B_k \in \Delta_n$ for some $k = 0, 1, ..., n$, Δ_n would not be compatible by Weakening and Cut for $|\!-^D$, which runs contrary to (1) above.

We may now show:

(3) $\Delta_n \,|\!-^D A_k(B_k)$ for $k = 0, 1, ..., n$.

Pf. There are three cases:

$B_k \in \Delta_n$ In this case, $A_k(B_k)$ is the formula $D^\infty(B_k)$. But $B_k \,|\!-^+ B_k^D$ and $B_k^D \,|\!-^+ \wedge(B_k^D) = D^\infty(B_k)$ by $\wedge$-Introduction. So by the structural rules for $|\!-^+$, $\Delta_n \,|\!- D^\infty(B_k)$.

$\neg B_k \in \Delta_n$ Similar to the previous case but with $\neg B_k$ in place of B_k.

$B_k, \neg B_k \notin \Delta_n$ In this case, $A_k(B_k)$ is the formula $I^\infty(B_k)$. Since $B_k, \neg B_k \notin \Delta_n$, it is clear from the construction that neither B_k nor $\neg B_k$ is compatible with Δ_k and hence, by (2), neither is compatible with Δ_n. So $\Delta_k, B_k \,|\!-^D C, \neg C$ for some formula C and $\Delta_k, \neg B_k \,|\!-^D C', \neg C'$ for some formula C′. By Reductio for $|\!-^D$, $\Delta_k \,|\!-^D \neg D^\infty(B_k)$ and $\Delta_k \,|\!-^D \neg D^\infty(\neg B_k)$; so by $\wedge$-Introduction, $\Delta_k \,|\!-^D I^\infty(B_k)$; and so by Weakening, $\Delta_n \,|\!-^D I^\infty(B_k)$.

Since Δ_n is compatible by (1) and $\Delta_n \,|\!-^D A_k(B_k)$ for $k = 0, 1, ..., n$ by (3), it follows that Δ_n is compatible with the response $A_0(B_0), A_1(B_1), ..., A_n(B_n)$ to $B_0, B_1, ..., B_n$ and hence so is the subset Δ_0 of Δ_n. It remains to show that the response $A_0(p), A_1(p), ..., A_n(p)$ is sharp. Since $B_1 \in \Delta_1$, $A_1(p) = D^\infty(p)$; and since $\neg B_n \in \Delta_1$, $A_n(p) = D^\infty(\neg p)$. This establishes the first condition for being a sharp response, viz.

that two of the individual responses should not be the same. Now the responses $A_0(p), A_1(p), ..., A_n(p)$ are of one of the three following forms: $D^\infty(p)$, $D^\infty(\neg p)$, and $I^\infty(p)$. The first two are truth-functionally inconsistent with the third (since $I^\infty(p)$ is the formula $\neg D^\infty p \wedge \neg D^\infty \neg p$) and the first two formulas are inconsistent with one another (since $D^\infty(p) \vdash p$ and $D^\infty(\neg p) \vdash \neg p$). This establishes the second condition for being a sharp response and we are done.

The collective response yielded by the proof of theorem contains three distinct individual responses - $D^\infty(p)$, $D^\infty(\neg p)$, and $I^\infty(p)$. But we may readily obtain a collective response that contains only two distinct individual responses by replacing the responses $D^\infty(p)$ or $D^\infty(\neg p)$, wherever they occur, with $\neg I^\infty(p)$. Call a response *bipartite* $A_0(p), A_1(p), ..., A_n(p)$ *bipartite* if there are exactly two formulas in the set $\{A_0(p), A_1(p), ..., A_n(p)\}$. We then have:
<u>Corollary</u> Take any formulas $B_0, B_1, ..., B_n$, $n > 0$. Then there is no set of formulas Δ_0 containing B_0 and $\neg B_n$ which is compatible and yet incompatible with any sharp bipartite response to $B_1, B_2, ..., B_n$.

In defining $\Delta \vdash^D A$, we have allowed ourselves to strengthen the premisses of Δ with arbitrary iterations of D's. But we might only allow ourselves to strengthen the premisses of Δ with a single iteration. Accordingly, let $\Delta \vdash^1 A$ hold if $D\Delta \vdash DA$; and let us say that Δ is *1-compatible* if there is no formula A for which $\Delta \vdash^1 A$ and $\Delta \vdash^1 \neg A$. Somewhat surprisingly, the impossibility theorem fails when $\vdash^1$ is substituted for $\vdash^D$.
<u>Theorem 2</u> Consider the sentence-letters $p_0, p_1, ..., p_n$, $n > 1$. Then there is a set of formulas Δ_0 containing p_0 and $\neg p_n$ which is 1-compatible and yet not 1-compatible with any sharp response to $p_1, p_2, ..., p_n$.
<u>Proof Sketch</u> We focus on the case n = 2, the proof for the general case being similar. Let $\vdash$ be the appropriate notion of consequence for the modal logic T. To construct Δ_0, we define four types of formula in the sentence-letter p (I have found it easier on the eye to revert to the standard notation for modal logic, with □ in place of D):

<u>Necessary Truth</u> NT(p) for $\Box p$;
<u>Necessary Falsehood</u> NF(p) for $\Box\neg p$;
<u>Contingent Truth</u> CT(p) for $p \wedge \neg\Box p$;
<u>Contingent Falsehood</u> CF(p) for $\neg p \wedge \neg\Box\neg p$.

We might, in the present context, call either of NT, NF, CT and CF a *modality*; and we say that the modality ψ is *consonant with* the modality ϕ (or that $\psi(p)$ is *consonant with* $\phi(p)$) if $\psi \neq \phi$ and either ϕ= NT and ψ = CT or ϕ = NF and ψ = CF or ϕ = CT or ϕ = CF. Intuitively, $\psi(p)$ is consonant with $\phi(p)$, for $\psi \neq \phi$, if the possibility of $\psi(p)$ is consistent with the truth of $\phi(p)$.

We now let:

$\Gamma_0 = \{\Box^n(NT(p_0) \wedge NF(p_2) \supset CT(p_1) \vee CT(p_1)): n = 1, 2, ...\} \cup \{\Box^n(\phi(p_k) \supset \Diamond\psi(p_k)): \psi$ is consonant with ϕ, k = 0, 1, 2, and n = 1, 2, ...$\}$; and

$\Delta_0 = \{p_0, \neg p_2\} \cup \Gamma_0$.

(Γ_0 has been so constructed that the failure to be □-closed arises solely from the presence of p_0 and $\neg p_2$.)

We first show that Δ_0 is 1-compatible, i.e. that there is a T-model M and world w of M such that $\Box\Delta_0$ is true at w in M. The model will be a tree-model in which the points are sequences of elements which get extended by one element in moving from a given point to an accessible point.

Each *element* will be a triple of formulas (E_0, E_1, E_2), where E_k, for k = 0, 1, 2 is one of the formulas NT(p_k), NF(p_k), CT(p_k) or CF(p_k) and it is not the case that: E_0 = NT(p_0), E_2 = NF(p_2) and either E_1 = NT(p_1) or E_1 = NF(p_1). Intuitively, the triple (E_0, E_1, E_2) is used to indicate that the formulas E_0, E_1, and E_2 are to be true at the given point. The element (F_0, F_1, F_2) is said to be *consonant with* the element (E_0, E_1, E_2) if, for k = 0, 1, 2, F_k is consonant with E_k. We take a *point* w to be a sequence $e_1, e_2, ..., e_n$ of elements, n > 0, for which e_{i+1} is consonant with e_i for i = 1, 2, ..., n-1; and we use w^L for the last element of the point w. The model $M_0 = (W_0, R_0, \upsilon_0)$ is then defined by:

W_0 = {w: w is a point};

R_0 = {(w, v): w, v $\in$ W and v = w or v = w⁀e for some element e};

υ_0 = {(w, p): w $\in$ W, p is p_k for some k = 0, 1, 2, and, for w^L = (E_0, E_1, E_2), E_k is either NT(p) or CT(p)}.

Let w_0 = (NT(p_0), CT(p_1), NF(p_2)). It is then readily verified that:

(1) $\Box\Delta_0$ is true at w_0 in M_0.

We now deal with the incompatibility condition. Given any model M = (W, R, ϕ) and world w of M, say that the *type of* p *at* w *in* M - Type(p, w, M) - is the modality ϕ for which ϕ(p) is true at w in M. Clearly, each world has exactly one type in a given sentence-letter.

(2) Suppose that M and N are generated models with respective base points w_0 and v_0, that Γ_0 is true at w_0 in M and at v_0 in N, that type(p_k, w, M) = type (p_l, v, N), for $0 \leq k, l \leq 2$, and that A(p) is a formula whose sole sentence letter is p. Then A(p_k) is true at w in M iff A(p_l) is true at v in N.

Proof By a straightforward formula induction.

Finally, suppose for reductio that $A_0(p_0)$, $A_1(p_1)$, $A_2(p_2)$ is a sharp response in p_0, p_1, p_2 and that it is 1-compatible with Δ_0. $\Box\Delta_0 \cup \{\Box A_0(p_0), \Box A_1(p_1), \Box A_2(p_2)\}$ is then true at some point w in a model M = (W, R, υ). We may show that $\Box A_1(p_0)$ is true at w. For take any v for which wRv. Since NT(p_0) and NF(p_2) are true at w, it follows that CT(p_1) or CF(p_1) is true at w. Without loss of generality, suppose that CT(p_1) is true at w. It should then be clear that for some u, wRu and type(p_1, u, M) = type(p_0, v, M). Since $\Box A_1(p_1)$ is true at w, $A_1(p_1)$ is true at u and so, by (2) above, $A_1(p_0)$ is true at v. But v was arbitrary, and so $\Box A_1(p_0)$ is true at w. It may similarly be shown that $\Box A_1(p_2)$. But then $A_0(p)$, and $A_1(p)$ are not inconsistent (indeed, not even incompatible) and so $A_0(p) = A_1(p)$. Similarly, $A_2(p) = A_1(p)$ and the response is not sharp after all.[5]

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