UNL - Department of Physics and Astronomy

Preliminary Examination - Day 1
Thursday, August 15, 2019

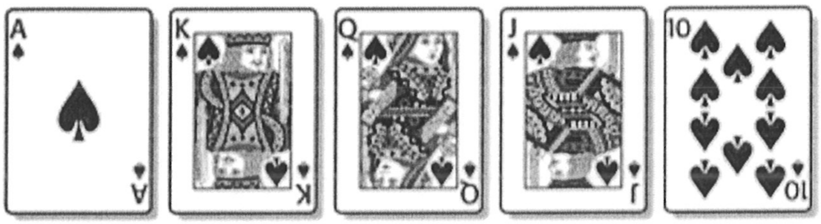
This test covers the topics of *Thermodynamics and Statistical Mechanics* (Topic 1) and *Quantum Mechanics* (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

Note: If you do more than two problems in a group, only the first two (in the order they appear in this handout) will be graded. For instance, if you do problems A1, A3, and A4, only A1 and A3 will be graded.

WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY

Thermodynamics and Statistical Mechanics Group A - Answer only two Group A questions

A1 Suppose you shuffle a standard deck of 52 cards and draw the top five cards from the deck. Then you repeat the procedure. Eventually, there is a chance that drawing the top five cards will yield the sequence ace through ten of spades in that order, as shown in the figure. Assuming that shuffling the deck thoroughly takes you about a minute, how long do you expect it to take, on average, before you will find this particular sequence of cards as the first five cards in the deck?



A2 An ideal monoatomic gas is compressed (no heat being added or removed in the process) so that its volume is halved. What is the ratio of the new pressure to the original pressure?

A3 What is the efficiency of the most efficient cyclic heat engine operating between heat reservoirs at temperature T_1 and T_2 where $T_1 > T_2$?

Give an example of a cycle on which maximally efficient ideal heat engine could be based.

A4 Using the differential form of the first law of thermodynamics, show that though U is a state function, the exchanged heat Q is not a state function.

Thermodynamics and Statistical Mechanics Group B - Answer only two Group B questions

B1 Consider the following ideal gas expansion cases:

a. Calculate the change in entropy when one mole of an ideal gas is allowed to expand freely into double its original volume.

b. What is the entropy change when one mole of each of two distinct non-interacting ideal gasses are allowed to mix, starting with equal volumes and temperatures?

c. What entropy change is there when the valve connecting two equal-volume and -temperature bulbs of the same gas is opened?

B2 A thin-walled vessel of volume V is filled with a gas of molecular mass m and is kept at constant temperature T. The gas slowly leaks out of the vessel through a hole of area A into surrounding vacuum. Find the time required for the pressure in the vessel to drop to $1/e$ of its original value.

Hint: the average speed of molecules at given temperature is given by: $\bar{v} = \sqrt{\frac{8k_BT}{\pi\, m}}$.

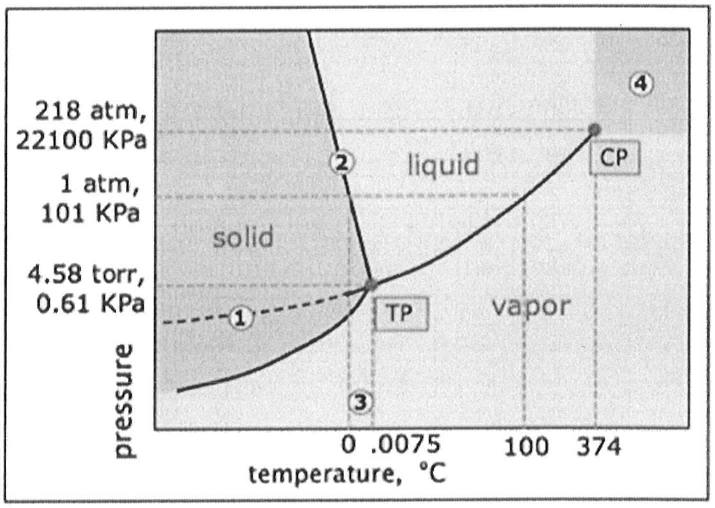
You may also need the expression for the differential of a solid angle: $d\Omega = \sin\theta\, d\theta d\varphi$.

B3 The phase diagram of water is shown below right on a log-log plot. This problem is focused on the melting transition (subscript "m"). The governing equation for the pressure dependence of this phase transition is given by the Clausius-Clapeyron equation:

$$\frac{dT_\mathrm{m}}{dP} = \frac{T_\mathrm{m}}{L_\mathrm{m}}\left(V_\mathrm{liq} - V_\mathrm{sol}\right).$$

a. From the phase diagram, what can you deduce about the sign of $(V_\mathrm{liq} - V_\mathrm{sol})$? What observations of water and ice support this?

b. Very close to the tricritical point, one may assume that L_m and $|V_\mathrm{liq} - V_\mathrm{sol}|$ are independent of pressure. Moreover, at the tricritical point $L_\mathrm{m} = 3.3 \times 10^5$ J/kg and $|V_\mathrm{liq} - V_\mathrm{sol}| = 9.0 \times 10^{-5}$ m³/kg. From this, find a relationship between T_m and P that is valid close to the tricritical point and describe its form (i.e. linear or quadratic or exponential or log or ...) (Hint: A Taylor expansion may be useful).



$\boxed{\textbf{B4}}$ Two identical bodies of constant heat capacity C_p are used as reservoirs for a heat engine. Their initial temperatures are T_1 and T_2, respectively. Assuming that the bodies remain at constant pressure and undergo no change of phase, derive the expression for the maximum work obtainable from the system.

Quantum Mechanics Group A - Answer only two Group A questions

A1 A typical "white dwarf" star is only about as big as the Earth, but with a surface temperature of 2.5×10^4 K. A typical "red giant" star has a surface temperature of 3.0×10^3 K and a radius about 1.0×10^5 times larger than that of a white dwarf.

a. Find the ratio of a white dwarf's total intensity (power per unit area, in watts/m^2, summed over all wavelengths) radiated from the surface to that of a red giant.
b. Find the ratio of the total power output (watts) of a white dwarf to that of a red giant.

A2 When light of a given wavelength is incident on a metallic surface, the stopping potential for the photoelectrons is 3.2 V. If a second light source whose wavelength is double that of the first is used, the stopping potential drops to 0.8 V. From this data, calculate

a. the wavelength of the first radiation;
b. the work function and the cutoff frequency of the metal.

A3 A beam of X-rays is scattered by electrons at rest. The wavelength of the X-rays scattered at 60° to the beam axis is 3.5 pm.

a. What is the incident photon energy?
b. What is the electron recoil energy?

A4

a. Find the frequency of radiation emitted by hydrogen atom for transition between the states with the principle quantum numbers n and $n-1$. Express it in terms of n and fundamental constants.

b. Consider an electron moving on a circular orbit in the Bohr atom and prove the Bohr's correspondence principle: in the limit of high n the frequency of revolution of the electron is the same as the frquency of radiation obtained in part *a*.

Quantum Mechanics Group B - Answer only two Group B questions

B1 Consider a system prepared at time $t=0$ in the state:

$$|\psi(t=0)\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ -1 \end{pmatrix}$$

In the 3-dimensional state space, two observables A and B have matrix representations

$$A = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

a. What is the probability that a measurement of A at time $t=0$ yields +2?
b. Suppose that, immediately afterward, B is measured. What is the probability that the value 0 will be obtained?

B2 Consider a particle of spin 1/2.

a. What are the eigenvalues and eigenvectors of the operator $S_x + S_y$?

Suppose a measurement of this operator is made, and the system is found to be in the state corresponding to the largest eigenvalue.

b. What is the probability that a measurement of S_z yields $\hbar/2$?
c. What is the probability that a measurement of S_x yields $\hbar/2$?

B3 A particle of mass m is moving in a harmonic field with frequency ω. At $t=0$ the particle is in the superposition state $\psi(x) = \frac{1}{\sqrt{2}}[\psi_0(x) + \psi_1(x)]$ where $\psi_n(x)$, $n = 0, 1, \ldots$ are the oscillator eigenstates.

a. Write the wave function at $t > 0$.
b. Find the expectation values of the energy and parity operator. Are they time-dependent? Explain why they are or are not.
c. Calculate the expectation value of the position coordinate x.
d. Calculate the expectation value of the momentum p.
e. Show that $\langle x \rangle$ and $\langle p \rangle$ satisfy the Heisenberg equation of motion.

B4 An electron is moving freely in a one-dimensional infinite potential box with walls at $x = 0$ and $x = a$. The electron is initially in the ground state ($n = 1$) of the box when the box suddenly quadruples in size (its right side instantly moving from $x = a$ to $x = 4a$. Calculate the probability of finding the electron in …

a. the ground state of the new box.

b. the first excited state of the new box.

Physical Constants

speed of light.................. $c = 2.998 \times 10^{8}$ m/s

Planck's constant .... $h = 6.626 \times 10^{-34}$ J·s

Planck's constant / 2π.... $\hbar = 1.055 \times 10^{-34}$ J·s

Boltzmann constant ...... $k_B = 1.381 \times 10^{-23}$ J/K

elementary charge .... $e = 1.602 \times 10^{-19}$ C

electric permittivity ...... $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m

magnetic permeability... $\mu_0 = 1.257 \times 10^{-6}$ H/m

molar gas constant .... $R = 8.314$ J / mol·K

Avogadro constant .... $N_A = 6.022 \times 10^{23}$ mol^{-1}

electrostatic constant ... $k = (4\pi\varepsilon_0)^{-1} = 8.988 \times 10^{9}$ m/F

electron mass .... $m_{el} = 9.109 \times 10^{-31}$ kg

electron rest energy ..... 511.0 keV

Compton wavelength .. $\lambda_C = h/m_{el}c = 2.426$ pm

proton mass .... $m_p = 1.673 \times 10^{-27}\,\text{kg} = 1836 m_{el}$

1 bohr............................ $a_0 = \hbar^2 / ke^2 m_{el} = 0.5292$ Å

1 hartree (= 2 rydberg) .... $E_h = \hbar^2 / m_{el} a_0^{\,2} = 27.21$ eV

gravitational constant ... $G = 6.674 \times 10^{-11}$ m^3 / kg s^2

hc .... $hc = 1240$ eV·nm

Equations That May Be Helpful

TRIGONOMETRY

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

$$\sin\alpha\sin\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$
$$\cos\alpha\cos\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$
$$\sin\alpha\cos\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$
$$\cos\alpha\sin\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$$

THERMODYNAMICS

Partition function = $Z = \Sigma_i e^{-\beta E_i}$

Average energy = $\langle E\rangle = -\frac{\partial}{\partial\beta}(\ln Z)$

Heat capacity = $C_V = N\frac{d\langle E\rangle}{dT}$

Clausius' theorem: $\sum_{i=1}^{N}\frac{Q_i}{T_i} \le 0$, which becomes $\sum_{i=1}^{N}\frac{Q_i}{T_i} = 0$ for a reversible cyclic process of N steps.

$$\frac{dp}{dT} = \frac{\lambda}{T\Delta V}$$

For adiabatic processes in an ideal gas with constant heat capacity, $pV^{\gamma} = \text{const.}$

$$dU = TdS - pdV$$

$$H = U + pV \qquad F = U - TS \qquad G = F + pV \qquad \Omega = F - \mu N$$

$$C_V = \left(\frac{\delta Q}{dT}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V \qquad C_p = \left(\frac{\delta Q}{dT}\right)_p = T\left(\frac{\partial S}{\partial T}\right)_p \qquad TdS = C_V dT + T\left(\frac{\partial S}{\partial V}\right)_T dV$$

$$\kappa = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T \qquad \alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p$$

Triple product: $\left(\frac{\partial X}{\partial Y}\right)_Z \cdot \left(\frac{\partial Y}{\partial Z}\right)_X \cdot \left(\frac{\partial Z}{\partial X}\right)_Y = -1$

Maxwell's relations:

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial p}{\partial S}\right)_V, \quad \left(\frac{\partial T}{\partial p}\right)_S = \left(\frac{\partial V}{\partial S}\right)_p, \quad \left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V, \quad -\left(\frac{\partial S}{\partial p}\right)_T = \left(\frac{\partial V}{\partial T}\right)_p$$

QUANTUM MECHANICS

Ground-state wavefunction of the hydrogen atom: $\psi(\mathbf{r}) = \dfrac{e^{-r/a_0}}{\pi^{1/2} a_0^{3/2}}$, where $a_0 = \dfrac{4\pi\varepsilon_0 \hbar^2}{me^2}$ is the Bohr radius, using $m \approx m_{\text{el}}$, in which m_{el} is the electron mass.

$$\psi_{nlm}(\mathbf{r}) = R_{nl}(r) Y_{lm}(\hat{\mathbf{r}}) \qquad E_n = -\frac{1}{n^2}\frac{mk^2 e^4}{2\hbar^2}$$

$$R_{10}(r) = \frac{2}{a_0^{3/2}} e^{-r/a_0}$$

$$R_{21}(r) = \frac{1}{3^{1/2}(2a_0)^{3/2}} \frac{r}{a_0} e^{-r/2a_0}$$

Particle in one-dimensional, infinitely-deep box with walls at $x = 0$ and $x = a$:

Stationary states $\psi_n = (2/a)^{1/2} \sin(n\pi x/a)$, energy levels $E_n = n^2 \dfrac{\pi^2\hbar^2}{2ma^2}$

Harmonic oscillator: $\psi_0(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2/2}$

$$\psi_1(x) = \frac{2^{1/2}\alpha^{3/4}}{\pi^{1/4}} x e^{-\alpha x^2/2} \qquad \text{where } \alpha = \frac{m\omega}{\hbar}$$

with $\langle \psi_0(x) | x | \psi_1(x) \rangle = (2\alpha)^{-1/2}$

Angular momentum: $[L_x, L_y] = i\hbar L_z$ *et cycl.*

Ladder operators:

$$L_+ | \ell, m\rangle = \hbar\sqrt{(\ell+m+1)(\ell-m)}\, | \ell, m+1\rangle$$
$$L_- | \ell, m\rangle = \hbar\sqrt{(\ell+m)(\ell-m+1)}\, | \ell, m-1\rangle$$

Creation, annihilation operators:

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right) \qquad \hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right)$$

$$\hat{a}^\dagger | n\rangle = \sqrt{n+1}\, | n+1\rangle \qquad \hat{a} | n\rangle = \sqrt{n}\, | n-1\rangle$$

Probability current density: $J(x) = \dfrac{\hbar}{2mi}\left(\psi^* \dfrac{\partial\psi}{\partial x} - \psi \dfrac{\partial\psi^*}{\partial x}\right) = \dfrac{\hbar}{m}\mathrm{Im}\left(\psi^* \dfrac{\partial\psi}{\partial x}\right)$.

Table Spherical harmonics and their expressions in Cartesian coordinates.

$Y_{lm}(\theta, \varphi)$	$Y_{lm}(x, y, z)$
$Y_{00}(\theta, \varphi) = \frac{1}{\sqrt{4\pi}}$	$Y_{00}(x, y, z) = \frac{1}{\sqrt{4\pi}}$
$Y_{10}(\theta, \varphi) = \sqrt{\frac{3}{4\pi}} \cos\theta$	$Y_{10}(x, y, z) = \sqrt{\frac{3}{4\pi}} \frac{z}{r}$
$Y_{1,\pm1}(\theta, \varphi) = \mp\sqrt{\frac{3}{8\pi}} e^{\pm i\varphi} \sin\theta$	$Y_{1,\pm1}(x, y, z) = \mp\sqrt{\frac{3}{8\pi}} \frac{x \pm iy}{r}$
$Y_{20}(\theta, \varphi) = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1)$	$Y_{20}(x, y, z) = \sqrt{\frac{5}{16\pi}} \frac{3z^2 - r^2}{r^2}$
$Y_{2,\pm1}(\theta, \varphi) = \mp\sqrt{\frac{15}{8\pi}} e^{\pm i\varphi} \sin\theta \cos\theta$	$Y_{2,\pm1}(x, y, z) = \mp\sqrt{\frac{15}{8\pi}} \frac{(x \pm iy)z}{r^2}$
$Y_{2,\pm2}(\theta, \varphi) = \sqrt{\frac{15}{32\pi}} e^{\pm 2i\varphi} \sin^2\theta$	$Y_{2,\pm2}(x, y, z) = \mp\sqrt{\frac{15}{32\pi}} \frac{x^2 - y^2 \pm 2ixy}{r^2}$

$$H_{\text{mag}} = -\gamma \mathbf{S} \cdot \mathbf{B}$$

Pauli matrices: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Compton scattering: $\lambda' - \lambda = \lambda_C(1 - \cos\theta)$

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl : $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

CARTESIAN AND SPHERICAL UNIT VECTORS

$$\hat{\mathbf{x}} = (\sin\theta\cos\phi)\hat{\mathbf{r}} + (\cos\theta\cos\phi)\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{y}} = (\sin\theta\sin\phi)\hat{\mathbf{r}} + (\cos\theta\sin\phi)\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

INTEGRALS

$f(x)$		$\int_0^\infty f(x)\,dx$
e^{-ax^2}	....	$\frac{\sqrt{\pi}}{2\sqrt{a}}$
xe^{-ax^2}	....	$\frac{1}{2a}$
$x^2 e^{-ax^2}$	....	$\frac{\sqrt{\pi}}{4a^{3/2}}$
$x^3 e^{-ax^2}$	....	$\frac{1}{2a^2}$
$x^4 e^{-ax^2}$	....	$\frac{3\sqrt{\pi}}{8a^{5/2}}$
$x^5 e^{-ax^2}$	....	$\frac{1}{a^3}$
$x^6 e^{-ax^2}$	....	$\frac{15\sqrt{\pi}}{16a^{7/2}}$

$$\int_0^\infty \frac{1}{1+bx^2}dx = \pi / 2b^{1/2}$$

$$\int_0^\infty x^n e^{-bx}dx = \frac{n!}{b^{n+1}}$$

$$\int (x^2+b^2)^{-1/2}dx = \ln\left(x+\sqrt{x^2+b^2}\right)$$

$$\int (x^2+b^2)^{-1}dx = \frac{1}{b}\arctan(x/b)$$

$$\int (x^2+b^2)^{-3/2}dx = \frac{x}{b^2\sqrt{x^2+b^2}}$$

$$\int (x^2+b^2)^{-2}dx = \frac{\frac{bx}{x^2+b^2}+\arctan(x/b)}{2b^3}$$

$$\int \frac{x\,dx}{x^2+b^2} = \frac{1}{2}\ln\left(x^2+b^2\right)$$

$$\int \frac{dx}{x(x^2+b^2)} = \frac{1}{2b^2}\ln\left(\frac{x^2}{x^2+b^2}\right)$$

$$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right)$$
$$= -\frac{1}{ab}\operatorname{artanh}\left(\frac{ax}{b}\right)$$