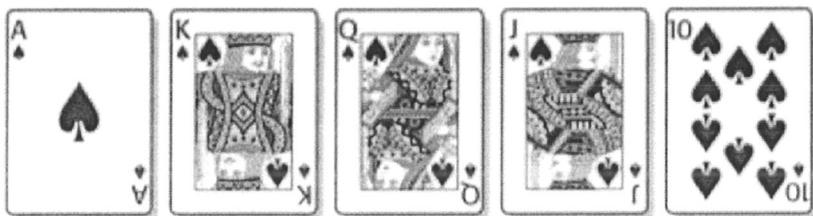


## Thermodynamics problems: SOLUTIONS

### Easy: Probability

Suppose you shuffle a standard deck of 52 cards and draw the top five cards from the deck. Then you repeat the procedure. Eventually, there is a chance that drawing the top five cards will yield the sequence ace through ten of spades in that order, as shown in the figure. Assuming that shuffling the deck thoroughly takes you about a minute, how long do you expect it to take, on average, before you will find this particular sequence of cards as the first five cards in the deck?



Each card has the probability  $\frac{1}{52}$  of being on top.  
Thus for the ace of spades the probability is  $\frac{1}{52}$ .  
After the ace of spades is drawn, 51 cards are left.  
The probability to draw the king of spades is  $\frac{1}{51}$ , Etc.  
Thus the probability to draw the specified 5 cards is

$$P_5 = \frac{1}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}$$

On average, the required number of shuffles  
is  $N = \frac{1}{P_5}$

and the average time it would take is

$$t = N \Delta t = \frac{\Delta t}{P_5} = 311\ 875\ 200 \text{ minutes}$$

$(593 \text{ years})$

### Easy: Adiabatic process

An ideal monoatomic gas is compressed (no heat being added or removed in the process) so that its volume is halved. What is the ratio of the new pressure to the original pressure?

The equation of state of an adiabatic process

is

$$P_1 V_1^\gamma = P_2 V_2^\gamma = \text{const}$$

where  $\gamma = 5/3$  for monoatomic gases.

As  $V_2 = \frac{1}{2} V_1$ , we can find

$$\frac{P_2}{P_1} = \left(\frac{V_1}{V_2}\right)^\gamma = 2^{5/3} \approx 3.17$$

### Easy: Heat engine/efficiency

What is the efficiency of the most efficient cyclic heat engine operating between heat reservoirs at temperature  $T_1$  and  $T_2$  where  $T_1 > T_2$ ?

The theoretical maximum efficiency of any heat engine depends only on the temperatures it operates between. This efficiency can be derived, for example, using a Carnot heat engine.

The efficiency is

$$\boxed{\eta = 1 - \frac{T_2}{T_1}}$$

(Exam takers should just know this, and won't need to really derive this. A simple answer in the box is ok for full credit.)

## Difficult: Entropy

Consider the following ideal gas expansion cases:

- (5 points) Calculate the change in entropy when one mole of an ideal gas is allowed to expand freely into double its original volume.
- (15 points) What is the entropy change when one mole of each of two distinct noninteracting ideal gasses are allowed to mix, starting with equal volumes and temperatures?
- (5 points) What entropy change is there when the valve connecting two equal-volume and -temperature bulbs of the same gas is opened?

a) The change in entropy is zero because the internal energy of an ideal gas is a function of temperature only. For free expansion of an ideal gas

$$T \Delta S = \Delta Q = \Delta U + \Delta W = 0 + 0 = 0$$

b) The entropy of one mole of an ideal gas at given T and p is

$$S(T, p) = C_p \ln T - R \ln p + K$$

where K is a numerical constant.

The entropy of a mixture of two ideal gases of one mole each, starting with equal volumes and temperature, is ,  $C_{p_1} = C_{p_2} = C_{p'} = C_p$  for an ideal gas.

$$S_{12}(T, p') = (C_{p_1} + C_{p_2}) \ln T - 2R \ln p' + 2K$$

where  $p'$  is the partial pressure of either of the gases. Since the volume occupied by each gas has doubled:

$$p' = \frac{1}{2} p$$

Over

The entropy change, then, is

$$S_{12} - 2S = 2R \ln p - 2R \ln \frac{P}{2} = \underline{\underline{2R \ln 2}}$$

c) In the case of mixture of same gas,  $p'$  in the final state equals  $p$  in the initial state. Therefore, the entropy change is zero.

## Difficult: Kinetic theory of gases

A thin-walled vessel of volume  $V$  is filled with a gas of molecular mass  $m$  and is kept at constant temperature  $T$ . The gas slowly leaks out of the vessel through a hole of area  $A$  into surrounding vacuum. Find the time required for the pressure in the vessel to drop to  $1/e$  of its original value. Hint: the average speed of molecules at given temperature is given by:

$$\bar{v} = \sqrt{\frac{8k_B T}{\pi m}}.$$

First, we find flux: the number of molecules crossing a unit surface area per unit of time:

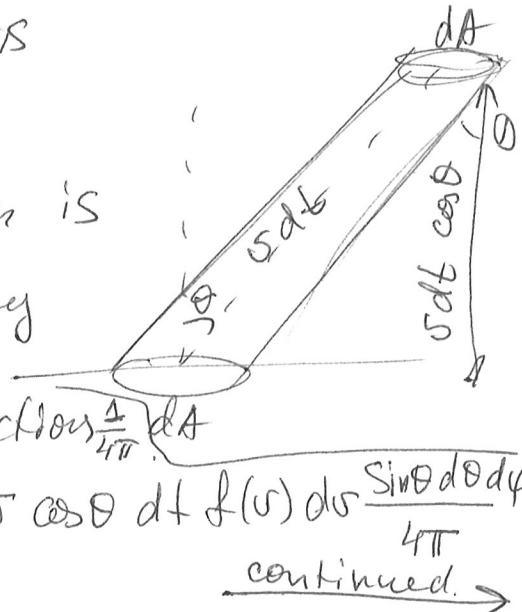
$$\Phi = \frac{dN}{dA dt}$$

Molecules are approaching  $dA$  from all directions  $(\theta, \phi)$  of a hemisphere. We will first consider  $dN(\theta, \phi)$  coming from a particular direction within the solid angle  $d\Omega$  around it. Out of these, we will consider only molecules with speeds between  $v$  and  $v + dv$ . The number of such molecules is the number within the short cylinder with base area  $dA$  and height  $(v \cos \theta dt)$ .

The volume of the cylinder is  $dV = dA v \cos \theta dt$ , and the number of molecules in it is

$$dN_V = \frac{N}{V} dV. \text{ From these, we pick molecules whose velocity magnitude is around } (v, \theta, \phi), \text{ using the probability distribution for speeds } f(v) \text{ and the uniform probability for the direction } \frac{1}{4\pi} d\Omega.$$

$$dN(v, \theta, \phi) = dN_V \cdot f(v) dv \cdot \frac{1}{4\pi} d\Omega = \frac{N}{V} dA v \cos \theta dt f(v) dv \frac{\sin \theta d\theta d\phi}{4\pi}$$



continued.

The flux for a given velocity magnitude and direction, then, is

$$\Phi(s, \theta, \phi) = \frac{dN(s, \theta, \phi)}{dA dt} = \frac{Ns \cos \theta f(s) dv}{V} \frac{\sin \theta d\theta d\phi}{4\pi}$$

The full flux through a unit surface area in one direction is obtained by integrating  $\Phi(s, \theta, \phi)$  over all speeds and over one hemisphere of angles:

$$\Phi = \int \frac{dN(s, \theta, \phi)}{dA dt} = \frac{N}{V \cdot 4\pi} \int_0^{\infty} s f(s) dv \cdot \int_0^{\pi/2} \sin \theta \cos \theta d\theta \int_0^{2\pi} d\phi.$$

$$= \frac{N}{V \cdot 4\pi} \bar{v} \cdot \pi = \frac{N \bar{v}}{4V}, \text{ where } \bar{v} \text{ is the average speed of molecules.}$$

For a given area  $A$ , the number of molecules leaving the vessel through  $A$  per unit of time is:

$$dN_A = \Phi A dt = \frac{N \bar{v} A}{4V} dt$$

and the number of molecules in the vessel obeys

$$\frac{dN}{dt} = -\frac{dN_A}{dt} \Rightarrow \frac{dN}{dt} = -\frac{N \bar{v} A}{4V}$$

thus

$$N(t) = N_0 e^{-\frac{A \bar{v} t}{4V}}$$

The pressure drops by  $\frac{1}{e}$  of the original when  $N$  drops to  $\frac{1}{e}$ , i.e.  $\frac{A \bar{v} t}{4V} = 1$

$$\Rightarrow \boxed{t = \frac{4V}{A \bar{v}}} = \boxed{\frac{4V}{A} \sqrt{\frac{\pi m'}{8k_B T}}}$$

### Difficult: Heat engine/work

Two identical bodies of constant heat capacity  $C_p$  are used as reservoirs for a heat engine. Their initial temperatures are  $T_1$  and  $T_2$ , respectively. Assuming the bodies remain at constant pressure and undergo no change of phase, derive the expression for the maximum work obtainable from the system.

Let  $t_1$  and  $t_2$  be the instantaneous temperatures of object 1 and object 2, and  $dW$  be the amount of work output from the system when  $dQ_1$  and  $dQ_2$  are added to each object, respectively. The maximum possible efficiency between two heat reservoirs at temperature  $t_1$  and  $t_2$  ( $t_1 > t_2$ ) is

$$\eta = \frac{t_1 - t_2}{t_1} = -\frac{dW}{dQ_1} = -\frac{dW}{C_p dt}$$

where the minus sign indicates that  $dQ_1$  is actually flowing out of object 1.

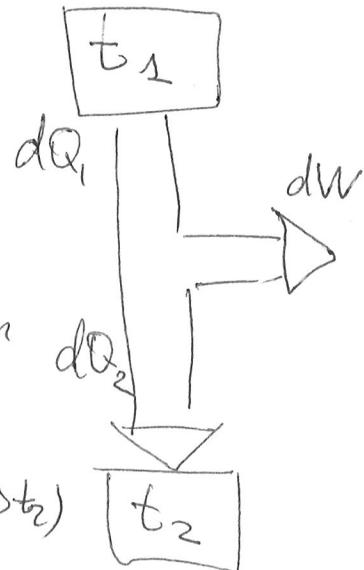
Therefore

$$-dW = C_p dt, \quad \frac{t_1 - t_2}{t_1} \quad (1)$$

According to the first law of thermodynamics the total energy flowing out of object 1 should be equal to the sum of the output work from the system and the heat flowing into object 2:

$$C_p (T_2 - t_1) = W + C_p (t_2 - T_2)$$

$$-C_p t_2 = W + C_p (t_1 - T_1 - T_2) \quad (2)$$



Substituting (2) into (1):

$$dW = -\frac{dt_1}{t_1} (W + C_p(2t_1 - T_1 - T_2))$$

$$t_1 dW = -W dt_1 - dt_1 \{C_p(2t_1 - T_1 - T_2)\}$$

$$d(Wt_1) = -C_p(2t_1 - T_1 - T_2) dt_1 \quad (3)$$

Let us denote  $T_f$  the final temperature  $T_f$ .

Integrating (3) from  $T_1$  to  $T_f$  we get

$$W \cdot T_f = C_p (T_1^2 - T_f^2 - (T_1 + T_2)(T_1 - T_f)) \quad (4)$$

(note:  $W=0$  when  $t_1=T_1$ )

The equation (2) is also valid for  $t_1=t_2=T_f$ ,  
which ~~gives~~ gives

$$-C_p T_f = W + C_p (T_f - T_1 - T_2) \quad (5)$$

$$(T_f = \frac{1}{2}(T_1 + T_2) - \frac{W}{C_p})$$

not needed.

We can now solve (4) and (5) to find  $W$ .

From (5)  $2C_p T_f = C_p (T_1 + T_2)W \Rightarrow W = C_p(T_1 + T_2) - 2C_p T_f$ .  
use that in (4)

$$[C_p(T_1 + T_2) - 2C_p T_f] T_f = C_p (T_1^2 - T_f^2 - (T_1 + T_2)(T_1 - T_f))$$

$$\Rightarrow T_f = \sqrt{T_1 T_2}$$

Substitute into (5)

$$\underline{W = C_p (T_1 + T_2 - 2\sqrt{T_1 T_2})}$$

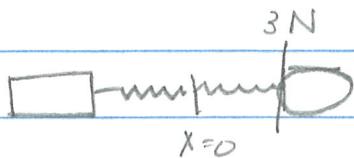
1

a) Position of block

$$F = -kx$$

$$\therefore 3 = 50x \Rightarrow x = \frac{3}{50} = 6 \text{ cm to the right.}$$

$$= \frac{6}{100} = 6 \times 10^{-2} \text{ m}$$



b) Work done on block by applied force

$$W = \int F \cdot dx = +3 \times .06 \quad F+dx \text{ are } \parallel$$

$$= 0.18 \text{ J}$$

c) Work done on block by spring force

Spring force is anti parallel to  $dx$ 

$$F = -kx \quad \therefore W = \int (-kx)(tdx) = -\frac{1}{2} kx^2 = -\frac{1}{2} 50(.06)^2$$

$$= -25 \times .06 \times 06 = -25 \times$$

$$= -0.09 \text{ J.}$$

d) Net work done ~~new~~ = 0.09 J

This is equal to the gain in P.E. of the block  $\frac{1}{2} kx^2 = +0.09 \text{ J}$

2.



Too easy?  
Too short?

At  $t = 0$   $I = 300 \text{ kg m}^2$

As child walks towards rim  $I = 300 + (12)r^2$

$$= 300 + (12)(0.8t)^2$$
$$I = 300 + (12)(0.64)t^2$$

From conservation of angular momentum  
(no external torque is applied)

$$\text{Limit } = L(t)$$

$$(300)(2) = [300 + (12)(0.64)t^2] \omega$$
$$\omega = \frac{600}{300 + 7.68t^2}$$

3

$$V(x) = \frac{A}{r^{12}} - \frac{B}{r^6}$$

a) Equilibrium position is when  $F=0$  i.e.  $-\frac{\partial V}{\partial r} = 0$

$$-\frac{12A}{r^{13}} + \frac{6B}{r^7} = 0 \quad ; \quad \frac{12A}{r^{13}} = \frac{6B}{r^7}$$

$$\therefore r_0^6 = 2A/B \quad B = \frac{2A}{r_0^6}$$

b) At  $r = r_0 + \delta r$   $V(x) = \frac{A}{r^{12}} - \frac{B}{r^6}$

$$F = -\frac{12A}{(r_0 + \delta r)^{13}} + \frac{6B}{(r_0 + \delta r)^7}$$

$$= -\frac{12A}{(r_0 + \delta r)^6} + \frac{6B}{(r_0 + \delta r)^6} = -\frac{12A}{(r_0 + \delta r)^6} + \frac{12A}{r_0^6}$$

Direction pulls towards equilibrium i.e towards  $r_0$  (inwards)

4. There should be a table for moment of inertia in the formula sheet.

a) The difference in the time arises from the different moments of inertia

The solid cylinder has a lower moment of inertia + hence higher linear K.E.  $\rightarrow$  Cylinder A

b) Cylinder A  $I = \frac{1}{2} M_A R^2$

$$M_A g h = \frac{1}{2} M_A V_A^2 + \frac{1}{2} \left( \frac{1}{2} M_A R^2 \right) \frac{V_A^2}{R^2} \quad \text{Rot K.E.} \\ = \frac{3}{4} V_A^2 \Rightarrow V_A^2 = \frac{4}{3} g h \quad \frac{m}{s^2}$$

vel at bottom

Cylinder B  $I = M_B R^2$

$$M_B g h = \frac{1}{2} M_B V_B^2 + \frac{1}{2} \left( M_B R^2 \right) \frac{V_B^2}{R^2}$$

$$g h = V_B^2 \quad \text{vel at bottom}$$

$$V_A - V_B = \sqrt{\frac{4}{3} g h} - \sqrt{g h}$$

c) No difference if they slide down plane

# THERMODYNAMICS.

a) The slope of the  $\frac{dP}{dT_m}$  line is negative

$$\therefore V_e < V_s \text{ i.e } P_e > P_s$$

We know this from observing that  
ice floats

b)  $\frac{dT_m}{dP} = T_m \left( V_{\text{lig}} - V_{\text{sol}} \right)$

$$= \frac{n L_m}{T_m} \left( 9 \times 10^{-5} \right) 273 \text{ K}$$

$$\approx 3.3 \times 10^5$$

of order  $1 - 10$

$\therefore \frac{dT_m}{dP}$  is very very small

$$\therefore P(T_m(P)) = P(T_m(\text{at } T_c)) + (T_m - T_{mc}) \left. \frac{dP}{dT} \right|_{T=T_c}$$

$$\therefore T_m(P) = \sim 273 \text{ K} + \Delta T \text{ (constant)}$$

$\therefore T_m(P)$  is linear in  $P$ .

**A4** Using the differential form of the first law of thermodynamics, show that though U is a state function, the exchanged heat Q is not a state function.

$$\text{The 1st Law : } dU = \underbrace{dQ}_{\substack{\text{heat} \\ \text{added} \\ \text{to system}}} + \underbrace{dW}_{\substack{\text{work done} \\ \text{on system}}} \quad \text{or } dQ - dW = \underbrace{dW}_{\substack{\text{work done} \\ \text{by system}}}$$

$$dQ = dU - dW$$

since  $U$  is a state function we can write (chain rule)

$$dU = \left(\frac{\partial U}{\partial P}\right)_V dP + \left(\frac{\partial U}{\partial V}\right)_P dV$$

$$\text{so } dQ = \cancel{\left(\frac{\partial U}{\partial P}\right)_V dP} + \left(\frac{\partial U}{\partial V}\right)_P dV - \underline{dW}$$

which we can  
write as  $dW = PdV$

$$dQ = \left(\frac{\partial U}{\partial P}\right)_V dP + \left[\left(\frac{\partial U}{\partial V}\right)_P + P\right] dV$$

Q M

Easy 1

$$\lambda' - \lambda = \lambda_c (1 - \cos 60^\circ) = \frac{\lambda_c}{2}$$

$$\lambda = \lambda' - \frac{\lambda_c}{2} = 3.5 - 1.2 = 2.3 \text{ pm}$$

(a)  $E = \frac{hc}{\lambda} = \frac{1240 \text{ keV} \cdot \text{pm}}{2.3 \text{ pm}} = 539 \text{ keV}$

(b)  $\frac{hc}{\lambda} = \frac{hc}{\lambda'} + E_e$

$$E_e = hc \left( \frac{1}{\lambda'} - \frac{1}{\lambda} \right) = 1240 \left( \frac{1}{2.3} - \frac{1}{3.5} \right) = 185 \text{ keV}$$

## QM Easy 2

(a) transition energy

$$\Delta E = -Ry \left( \frac{1}{n^2} - \frac{1}{(n-1)^2} \right) = Ry \frac{2n-1}{n^2(n-1)^2}$$

$$\omega = \frac{\Delta E}{\hbar} = \frac{m e^2 c^4}{2 \hbar^3} \frac{2n-1}{n^2(n-1)^2}$$

$$\text{for } n \geq 1 \quad \omega = \frac{m e^2 c^4}{\hbar^3 n^3} \quad (1)$$

(b) Bohr quantization rule

$$mv r = n\hbar$$

$$\text{or} \quad mv r^2 = n\hbar \quad r = a_0 n^2$$

$$\omega = \frac{n\hbar}{mr^2} = \frac{n\hbar}{m a_0^2 n^4} = \frac{\hbar}{m a_0^2 n^3}$$

Using  $a_0 = \frac{\hbar^2}{m e^2 c^2}$ , we obtain  $\omega = \frac{m e^2 c^4}{\hbar^3 n^3}$   
agrees with (1)

QM hard 1

$$S_x + S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & 1-i \\ 1+i & 0 \end{pmatrix}$$

$$(a) \quad \begin{vmatrix} -\lambda & 1-i \\ 1+i & -\lambda \end{vmatrix} = 0 \rightarrow \lambda^2 - 2 = 0 \quad \lambda = \pm \sqrt{2}$$

$$\text{eigenvalues } \pm \frac{\hbar\sqrt{2}}{2}$$

$$\text{eigenvectors: } \begin{pmatrix} a \\ 0 \end{pmatrix} \text{ where } b = \frac{\lambda}{\hbar i} a$$

For the higher eigenvalue

$$\chi_1 = N \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1-i}{\sqrt{2}} \end{pmatrix} \quad \text{For lower } \chi_2 = N \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{pmatrix}$$

$$N^2 \left( 1 + \frac{2}{2} \right) = 1 \quad N = \frac{1}{\sqrt{2}}$$

(b) prob. amplitude

$$\langle \chi_1 | S_z = \frac{\hbar}{2} \rangle = \frac{1}{\sqrt{2}} \left( 1, \frac{\sqrt{2}}{1+i} \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \quad P = \frac{1}{2} (50\%)$$

(c)

$$\langle \chi_1 | S_x = \frac{\hbar}{2} \rangle = \frac{1}{\sqrt{2}} \left( 1, \frac{\sqrt{2}}{1+i} \right) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} \left( 1 + \frac{\sqrt{2}}{1+i} \right) = \frac{1}{2} \frac{1+i\sqrt{2}+i}{1+i}$$

$$P = \frac{1}{4} \left| \frac{1+i\sqrt{2}+i}{1+i} \right|^2 = \frac{1}{4} \frac{1+2+i2\sqrt{2}+1}{2} = \frac{1}{2} \left( 1 + \frac{\sqrt{2}}{2} \right) = 0.85 (85\%)$$

## QM hard 2

$$(a) \psi(x, t) = \frac{1}{\sqrt{2}} [\psi_0(x) + e^{-i\omega t} \psi_1(x)]$$

(dropping the phase factor  $e^{-i\omega t/2}$ )

$$(b) E = \frac{1}{2} \int (\psi_0^* + e^{i\omega t} \psi_1^*) H (\psi_0 + e^{-i\omega t} \psi_1) dx$$

$$= \frac{1}{2} (E_0 + E_1) \quad \text{where } E_n = \hbar\omega(n + \frac{1}{2})$$

$$\langle P \rangle = \frac{1}{2} \int (\psi_0^* + e^{i\omega t} \psi_1^*) (\psi_0 - e^{-i\omega t} \psi_1) dx$$

(  $\psi_0$  is even,  $\psi_1$  is odd )

$$= \frac{1}{2} \int (|\psi_0|^2 - |\psi_1|^2) dx = 0$$

both values are time independent because  
 $H$  is independent on time and  $[H, P] = 0$

$$(c) \langle x \rangle = \frac{1}{2} \int (\psi_0^* + e^{i\omega t} \psi_1^*) x (\psi_0 + e^{-i\omega t} \psi_1) dx$$

$$= \frac{1}{2} \left[ e^{i\omega t} \int \psi_1^* x \psi_0 dx + e^{-i\omega t} \int \psi_0^* x \psi_1 dx \right]$$

$$= \text{Re} \left[ e^{-i\omega t} \int \psi_1^* x \psi_0 dx \right] = \sqrt{\frac{\hbar}{2m\omega}} \cos \omega t$$

$$(d) \langle p \rangle = \frac{1}{2} \int (\psi_0^* + e^{i\omega t} \psi_1^*) p (\psi_0 + e^{-i\omega t} \psi_1) dx$$

$$= \frac{1}{2} \left[ e^{i\omega t} \int \psi_1^* p \psi_0 dx + e^{-i\omega t} \int p^* \psi_0^* \psi_1 dx \right]$$

( Since  $p$  is Hermitian  $\langle \psi_1 | p | \psi_0 \rangle = \langle p \psi_0 | \psi_1 \rangle$  )

$$= -\frac{i\hbar}{2} \left[ e^{i\omega t} \int \psi_1^* \frac{\partial \psi_0}{\partial x} dx - e^{-i\omega t} \int \frac{\partial \psi_0^*}{\partial x} \psi_1 dx \right]$$

Q M hand 2 continued

$$\langle p \rangle = \hbar \operatorname{Im} [e^{i\omega t} \int \psi^* \frac{\partial \psi}{\partial x} dx]$$

$$\psi_0 = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2/\hbar} \quad \alpha = \frac{m\omega}{\hbar}$$

$$\frac{d\psi_0}{dx} = -\alpha x \psi_0 \quad , \text{ therefore}$$

$$\int \psi^* \frac{\partial \psi_0}{\partial x} dx = -\alpha \int \psi^* x \psi_0 dx = -\alpha \frac{1}{(2\alpha)^n} = -\left(\frac{\alpha}{2}\right)^n$$

$$\text{and } \langle p \rangle = -\hbar \sin \omega t \left(\frac{m\omega}{2\hbar}\right)^n = -\left(\frac{m\omega \hbar}{2}\right)^n \sin \omega t \quad (1)$$

(e) Heisenberg eq. of motion

$$\frac{d\langle x \rangle}{dt} = \frac{\langle p \rangle}{m} \quad (2)$$

$$\text{From (c)} \quad \frac{d\langle x \rangle}{dt} = -\left(\frac{\hbar \omega}{2m}\right)^n \sin \omega t$$

agrees with (1) and (2)

## Solution - State

- (a) Need to find the eigenstates and eigenvalues of  $\hat{A}$ .

Eigenvalues are  $\lambda = 0, \pm 2$  corresponding to the eigenstates

$$|A=0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$|A=\pm 2\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ \pm \sqrt{2} \\ 1 \end{pmatrix}$$

Thus, The probability of measuring  $A = +2$  for the state  $|4\rangle$  is

$$\begin{aligned} |\langle A=+2 | 4 \rangle|^2 &= \left| \frac{1}{2} (1 + \sqrt{2}) \begin{pmatrix} 1 \\ \sqrt{2} \\ -1 \end{pmatrix} \right|^2 \\ &= \left| \frac{1}{4} (1+2-1) \right|^2 = + \left| \frac{1}{2} \right|^2 = \frac{1}{4} \end{aligned}$$

or 25%

- (b) Now the system is in the state  $\frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$ .

Hence the probability of measuring  $B = +0$

$$\text{is } \left| \frac{1}{2} (1 - \sqrt{2}) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right|^2 = \left| \frac{\sqrt{2}}{2} \right|^2 = \frac{1}{2} \text{ or } 50\%.$$

**B4** An electron is moving freely in a one dimensional infinite potential box withg walls at  $x = 0$  and  $x = a$ . The electron is initially in the ground state ( $n = 1$ ) of the box when the box suddenly quadruples in size (it's right side instantly moving from  $x = a$  to  $x = 4a$ ). Calculate the probability of finding the electron in

- (a) the ground state of the new box.  
 (b) The first excited state of the new box.

Initially  $E_1 = \frac{-\pi^2 \hbar^2}{2ma^2}$  and  $\phi_1(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right)$

The new ground state energy is  $E'_1 = \frac{-\pi^2 \hbar^2}{2m(4a)^2} = \frac{-\pi^2 \hbar^2}{32ma^2}$   
 with wave function  $\psi_1(x) = \frac{1}{\sqrt{2a}} \sin\left(\frac{\pi x}{4a}\right)$

a)  $P(E'_1) = |\langle \psi_1 | \phi_1 \rangle|^2 = \left| \int_0^a \psi_1^*(x) \phi_1(x) dx \right|^2 = \frac{1}{a^2} \left| \int_0^a \sin\left(\frac{\pi x}{4a}\right) \sin\left(\frac{\pi x}{a}\right) dx \right|^2$

Initially  $\phi_1(x)$  does not exist beyond  $x=a$  use:  $\sin a \sin b = \frac{1}{2} \cos(a-b) - \frac{1}{2} \cos(a+b)$

$$\begin{aligned} P(E'_1) &= \frac{1}{a^2} \left| \frac{1}{2} \int_0^a \cos\left(\frac{3\pi x}{4a}\right) dx - \frac{1}{2} \int_0^a \cos\left(\frac{5\pi x}{4a}\right) dx \right|^2 \\ &= \frac{128}{15^2 \pi^2} = 0.058 = 5.8\% \end{aligned}$$

b) In the new box  $E'_2 = \frac{-\pi^2 \hbar^2}{8ma^2}$ ,  $\psi_2(x) = \frac{1}{\sqrt{2a}} \sin\left(\frac{\pi x}{2a}\right)$

$$\begin{aligned} \text{so } P(E'_2) &= |\langle \psi_2 | \phi_1 \rangle|^2 = \left| \int_0^a \psi_2^*(x) \phi_1(x) dx \right|^2 \\ &= \frac{1}{a^2} \left| \int_0^a \sin\left(\frac{\pi x}{2a}\right) \sin\left(\frac{\pi x}{a}\right) dx \right|^2 \\ &= \frac{16}{9\pi^2} = 0.18 = 18\% \end{aligned}$$

To obtain  $\lambda_1$  we have only to subtract (1.163) from (1.162):

$$V_{s_1} - V_{s_2} = \frac{hc}{e\lambda_1} \left(1 - \frac{1}{2}\right) = \frac{hc}{2e\lambda_1}. \quad (1.164)$$

The wavelength is thus given by

$$\lambda_1 = \frac{hc}{2e(V_{s_1} - V_{s_2})} = \frac{6.6 \times 10^{-34} \text{ Js} \times 3 \times 10^8 \text{ m s}^{-1}}{2 \times 1.6 \times 10^{-19} \text{ C} \times (3.2 \text{ V} - 0.8 \text{ V})} = 2.6 \times 10^{-6} \text{ m}. \quad (1.165)$$

(b) to obtain the work function, we simply need to multiply (1.163) by 2 and subtract the result from (1.162),  $V_{s_1} - 2V_{s_2} = W/e$ , which leads to

$$W = e(V_{s_1} - 2V_{s_2}) = 2.4 \text{ eV} = 2.4 \times 1.6 \times 10^{-19} = 3.84 \times 10^{-19} \text{ J}. \quad (1.166)$$

The cutoff frequency is

$$\nu = \frac{W}{h} = \frac{3.84 \times 10^{-19} \text{ J}}{6.6 \times 10^{-34} \text{ Js}} = 5.8 \times 10^{14} \text{ Hz}. \quad (1.167)$$

the spacings between them. Although the quantum effects are negligible for macroscopic systems, they are important for atomic and molecular systems beyond human detection. So quantum effects are negligible for macroscopic systems.

### Problem 1.3

When light of a given wavelength is incident on a metallic surface, the stopping potential for the photoelectrons is 3.2 V. If a second light source whose wavelength is double that of the first is used, the stopping potential drops to 0.8 V. From this data, calculate

- (a) the wavelength of the first radiation
- (b) the work function and the cutoff frequency of the metal

### Solution

(a) Using (1.22) and since the wavelength of the second radiation is double that of the first one,  $\lambda_2 = 2\lambda_1$ , we can write

$$V_{s1} = \frac{hc}{e\lambda_1} - \frac{W}{e}, \quad (1.162)$$

$$V_{s2} = \frac{hc}{e\lambda_2} - \frac{W}{e} = \frac{hc}{2e\lambda_1} - \frac{W}{e}. \quad (1.163)$$