TH A1

The probability that a single ¹⁰C hasn't decayed after *t* seconds is $p(t) = (\frac{1}{2})^{t/T}$, where *T* is the half-life. At time 30 s, this probability is $p(t = 30) = (\frac{1}{2})^{30/20} = 0.35355 = 35.4\%$.

The probability for "has decayed" is then 100% - 35.4% = 64.6%. The answer is therefore $P = \begin{pmatrix} 13 \\ 4 \end{pmatrix} p^4 (1-p)^9 = \frac{13!}{4!9!} (0.354)^4 (0.646)^9 = 22.0\%$, where the binomial coefficient accounts for the indistinguishability of the 13 nuclei.

TH B1

To find the entropy S(V,T) use

$$TdS = dU + pdV$$
$$dS = \frac{\partial S}{\partial T}\Big|_{V} dT + \frac{\partial S}{\partial V}\Big|_{T} dV \qquad dU = \frac{\partial U}{\partial T}\Big|_{V} dT + \frac{\partial U}{\partial V}\Big|_{T} dV$$

Then

$$\frac{\partial S}{\partial T}\Big|_{V} = \frac{1}{T} \frac{\partial U}{\partial T}\Big|_{V} = \frac{f}{2} \frac{Nk_{B}}{T} \text{ or } S \sim \frac{f}{2} Nk_{B} \ln T$$

$$\frac{\partial S}{\partial V}\Big|_{T} = \frac{1}{T} \left(\frac{\partial U}{\partial V} \Big|_{T} + P \right) = \frac{Nk_{B}}{V} \text{ or } S \sim Nk_{B} \ln V$$

Thus
$$S \sim Nk_B \ln\{T^{f/2}V\}$$

B) For an adiabatic process $dS = 0 \Rightarrow T^{f/2}V = \text{constant}$

C)
$$dQ = dU + pdV = \frac{\partial U}{\partial T}\Big|_{V} dT + \left(\frac{\partial U}{\partial V}\Big|_{T} + p\right) dV$$

 $\frac{dQ}{dT}\Big|_{V} = \frac{\partial U}{\partial T}\Big|_{V} = \frac{f}{2}Nk_{B} \quad \text{and} \quad C_{V} = \frac{f}{2}k_{B}$
 $\frac{dQ}{dT}\Big|_{V} = \frac{\partial U}{\partial T}\Big|_{V} + \left(\frac{\partial U}{\partial V}\Big|_{T} + p\right)\frac{\partial V}{\partial T}\Big|_{P} = \frac{\partial U}{\partial T}\Big|_{V} + \left(\frac{\partial U}{\partial V}\Big|_{T} + p\right)\cdot\left(-\left(\frac{\frac{\partial p}{\partial T}}{\frac{\partial p}{\partial V}}\right)\right)$

Substituting

$$\frac{\partial p}{\partial T}\Big|_{V} = \frac{Nk_{B}}{V} \qquad \& \qquad \frac{\partial p}{\partial V}\Big|_{T} = -\frac{Nk_{B}T}{V^{2}} + 2b\frac{N^{2}}{V^{3}}$$

gives
$$C_P = C_V + \frac{Nk_B^2 TV}{Nk_B TV - 2b_2 N^2} = \frac{f}{2}k_B + \frac{k_B}{1 - 2b\frac{N/V}{k_B T}}$$

where the following triple product was used:

$$\left(\frac{\partial V}{\partial T}\right)_{p}\left(\frac{\partial p}{\partial V}\right)_{T}\left(\frac{\partial T}{\partial p}\right)_{V} = -1 \quad \Rightarrow \quad \left(\frac{\partial V}{\partial T}\right)_{p} = -\frac{1}{\left(\frac{\partial p}{\partial V}\right)_{T}\left(\frac{\partial T}{\partial p}\right)_{V}} = -\frac{\left(\frac{\partial p}{\partial T}\right)_{V}}{\left(\frac{\partial p}{\partial V}\right)_{T}}$$

Thermodynamics Solutions

TH A2 Problem; First Law (every) Head released in isothermal process. Because the energy of an ideal gas depends only on the temperature, there is no change in its internal energy for an isothermal process. Therefore, according to the first law! $\Delta E = O = Q + W \Rightarrow Q = -W$ $W_{1\rightarrow 2} = -\int_{V_1} P(V,T) dV = -\int_{V_1} \frac{nRT}{V} dV = -nRT \ln \frac{V_2}{V_1}$ dW=-PdV V2 $Q = -W = hRT lu \frac{V_2}{V_1}$ Problem: Max well relations (easy) TH A3 Start with the full differential for internal energy: dU = TdS - PdV (formula sheef) It follows that $T = \left(\frac{\partial U}{\partial S}\right)_{V}$ and $P = -\left(\frac{\partial E}{\partial V}\right)_{e}$ For the second derivatives, we have: $\frac{\partial^2 U}{\partial v \partial s} = \frac{\partial^2 U}{\partial s \partial v}$, or $\left(\frac{\partial T}{\partial V}\right)_{S} = -\left(\frac{\partial P'}{\partial S}\right)_{V}$

Problem: heat engines, cycles (easy)

$$r_{1} + \frac{4}{a} + \frac{5}{b} = The energy the working Substance
absorbs from the source of high
$$r_{2} + \frac{5}{d} = C + temperature is$$

$$V = R_{ab} = Cp(T_{b} - T_{a})$$
The energy it gives to the source of lower
temperature is $R_{out} = O_{cd} = Cp(T_{c} - T_{d})$
The efficiency is:

$$D = 1 - \frac{R_{out}}{Q_{iu}} = 1 - \frac{T_{c} - T_{d}}{T_{b} - T_{a}}$$
Use equation of state $pV = nRT$:

$$D = 1 - \frac{P_{2}(V_{c} - V_{d})}{V_{i}(V_{b} - V_{a})}$$
Use a diabatic equation.

$$P_{1}V_{a}^{S} = P_{2}V_{d}, \qquad P_{1}V_{b}^{S} - P_{2}V_{c}^{S} + \frac{C}{P_{1}}V_{c}^{S}$$

$$D = 1 - \frac{P_{2}(V_{c} - V_{d})}{V_{i}(V_{b} - V_{a})}$$

$$D = 1 - \frac{P_{2}(V_{c} - V_{d})}{V_{i}(V_{b} - V_{a})} = 1 - \frac{P_{2}(V_{c} - V_{c})}{P_{1}(V_{b} - V_{a})}$$$$

Problem: raindrops (difficult)
We split the square into regions I
and I.
Consider region **T**. Here,
Min
$$(\mathcal{Y}, \mathcal{X}, \mathcal{Y}) = \mathcal{X}$$

Probability to find a drop between \mathcal{X} and \mathcal{X} the
if the drop is in region I, is $dP = \frac{(1-\mathcal{X})d\mathcal{X}}{S_{\mathrm{I}}}$
Where $S_{\mathrm{I}} = 1/2$, the area of region I.
The probability dousity is
 $P(\mathcal{X}) = 2(1-\mathcal{X})$
 $\mathcal{X} = \int P(\mathcal{X}) \mathcal{X} \mathcal{X} = \int 2(1-\mathcal{X}) \mathcal{X} \mathcal{X} = 2(\frac{\mathcal{X}^2 - \mathcal{X}^3}{2})|_{0}^{-1}$
Similarly for region I; interchauge $\mathcal{X} \ll \mathcal{Y}$
 $\overline{\mathcal{Y}_{\mathrm{II}}} = \frac{1}{3}$
In each of the regions I, I the average
of min(\mathcal{X}, \mathcal{Y}) is $\overline{\mathcal{X}_{\mathrm{I}}}$, which is the answer
we seek.

problem; two fluids, entropy, Carnot, Frestlaw (hard
Since
$$C_{1} = T\left(\frac{\partial S}{\partial T}\right)_{r}$$
,
the entropy lost by F_{1} is
 $S_{1} = -C_{1}\int_{0}^{\infty} dT = C_{1}\ln T_{1}$
the entropy gained by F_{2} is then,
 $S_{2} = C_{2}\ln T_{2}$
If the Carnot engine ends in the same state as it
started from, $\Delta S = 0$. Therefore:
 $S_{2} - S_{1} = \ln \left[\left(\frac{T_{0}}{T_{2}} \right)^{c_{2}} \left(\frac{T_{0}}{T_{1}} \right)^{c_{1}} \right] = 0$
 $\int \frac{V}{T_{0}} = T_{1} \frac{C_{1}}{C_{1}C_{2}} T_{2}^{c_{2}} \frac{d}{T_{0}}$
The internal energy lost by F_{1} is in the form
of heat
 $R_{1} = -C_{1} \int T_{0} dT = C_{1}(T_{1} - T_{0})$
The energy gained by F_{2} is
 $R_{2} = C_{2} (T_{0} - T_{2})$
The observation of energy, this hunst be
equal to the total work by the Carnot engine.
 $W = C_{1}T_{1} + C_{2}T_{2} - (C_{1} + C_{2})T_{0}$

Problem: Entropy in a paramagnetic system (herd)
a)
$$dR = dU + dW = -d(MB) + M dB = -B dM$$

b) $dS = \frac{dR}{T} = -\frac{B dM}{T} = -\frac{M}{C} dM$
c) Integrate dS over M, we get:
 $S = S_0 - \frac{1}{2C} M^2$

CM A1 KU

ANSWER

(MasteringPhysics 7.38)

IDENTIFY: For the system of two blocks, only gravity does work. Apply $K_1 + U_1 = K_2 + U_2$. **SET UP:** Call the blocks *A* and *B*, where *A* is the more massive one. $v_{A1} = v_{B1} = 0$. Let y = 0 for each block to be at the initial height of that block, so $y_{A1} = y_{B1} = 0$. $y_{A2} = -1.20$ m and $y_{B2} = +1.20$ m. $v_{A2} = v_{B2} = v_2 = 3.00$ m/s. **EXECUTE:** $K_1 + U_1 = K_2 + U_2$ gives $0 = \frac{1}{2}(m_A + m_B)v_2^2 + g(1.20 \text{ m})(m_B - m_A)$, with $m_A + m_B = 22.0$ kg.

Therefore $\frac{1}{2}(22.0 \text{ kg})(3.00 \text{ m/s})^2 + (9.80 \text{ m/s}^2)(1.20 \text{ m})(22.0 \text{ kg} - 2m_A)$. Solving for m_A gives

 $m_A = 15.2$ kg. And then $m_B = 6.79$ kg.

EVALUATE: The final kinetic energy of the two blocks is 99 J. The potential energy of block A decreases by 179 J. The potential energy of block B increases by 80 J. The total decrease in potential energy is 179 J – 80 J = 99 J, which equals the increase in kinetic energy of the system.

CM A2 KU

$$K = \frac{1}{2}mv^{2} + \frac{1}{2}I\omega^{2} = \frac{1}{2}mv^{2} + \frac{1}{2}(\beta mR^{2})\left(\frac{v}{R}\right)^{2} = \frac{1}{2}mv^{2} + \beta\frac{1}{2}mv^{2} \quad \text{since } \omega = v/R$$

$$f = \frac{\beta\frac{1}{2}mv^{2}}{\frac{1}{2}mv^{2} + \beta\frac{1}{2}mv^{2}} = \frac{\beta}{1+\beta}$$
cylinder: $\beta = \frac{1}{2} \implies f = \frac{\frac{1}{2}}{1+\frac{1}{2}} = \frac{1}{3} = \frac{\frac{35}{105}}{\frac{105}{105}}$
solid sphere: $\beta = \frac{2}{5} \implies f = \frac{\frac{2}{5}}{1+\frac{2}{5}} = \frac{2}{7} = \frac{\frac{30}{105}}{\frac{105}{105}}$
hollow sph.: $\beta = \frac{2}{3} \implies f = \frac{\frac{2}{3}}{1+\frac{2}{3}} = \frac{2}{5} = \frac{\frac{42}{105}}{\frac{105}{105}} \leftarrow \text{ greatest}$

August 2016

MECHANICS

CM A3 TG





• $W_f^2 - W_i^2 = 2\alpha \Delta \Theta$ 100 * (0.1047 red) - 0 = 2 × 10 rev × 2 Trad X $\Rightarrow \alpha = \frac{109.7 \frac{r_2 d^2}{5^2}}{125.7 r_2 d} = 0.873 \frac{r_2 d}{5^2}$

 $\omega = \alpha t$ (U(1.35s) = 1.18 red s $\partial_{\text{cent}} = V_{tah}^2 / R = \omega^2 R = 4.17 \, m/2^2$ $2_{tan} = \alpha R = 2.62 m/s^2$ 2 = V22 + 22 = 4.92 M/52

CM A4 TG

August 2016



mc = 640 acres $1 mc = 1.6 \times 10^3 m$ 3ft = 0.91m

 $(27 \text{ acre-ft}) \times \left(\frac{1 \text{ mi}}{640 \text{ acres}}\right) \times \left(\frac{1.6 \times 10^{6} \text{ m}^{2}}{1 \text{ mi}^{2}}\right) \times \left(\frac{0.91 \text{ m}}{3 \text{ ft}}\right)$

 $= 3.28 \times 10^4 \text{ m}^3$ This is the equivalent of 3.28×107 kg of HzO P = W/time

 $= \frac{mgh}{T} = \frac{3.28 \times 10^7 k_g \times 9.8 M_5^2 \times 340 m}{24 \times 3600 s}$

= /1.26 MW |

CM B1 SL

A mass *m* moves in a circular orbit of radius r_0 under the influence of a central force whose potential is $-\frac{km}{r^n}$. Show that the circular orbit is stable under small oscillations (that is, the mass will oscillate about the circular orbit) if n < 2.

CM B1 SL

8. For a particle moving under the influence of a central force, the effective potential is:

$$V_{
m eff} = V(r) + rac{L^2}{2mr^2}$$

A circular orbit is possible at that value of r for which $\partial V_{eff}/\partial r = 0$; and the orbit is stable if $\partial^2 V_{eff}/\partial r^2$ is positive. For this problem,

$$V_{\text{eff}} = -\frac{km}{r^n} + \frac{L^2}{2mr^2}, \qquad \frac{\partial V_{\text{eff}}}{\partial r} = \frac{kmn}{r^{n+1}} - \frac{L^2}{mr^3},$$

and

$$\frac{\partial^2 V_{\text{eff}}}{\partial r^2} = -\frac{kmn(n+1)}{r^{(n+2)}} + \frac{3L^2}{mr^4}.$$

Setting $\partial V_{\text{eff}}/\partial r = 0$, one finds that $\partial^2 V_{\text{eff}}/\partial r^2 > 0$ if 3 - (n + 1) > 0, i.e. n < 2.

Linear Motion of a triatomic molecule: A carbon dioxide molecule CO_2 , which has the structure as shown in figure. We consider motion only in one dimension, along the *x*-axis. The two end particle each of mass *m*, are bound to the central particle, mass *M*, via a potential function that is equivalent to that of two springs of stiffness *K*. The coordinates expressing the displacements of each mass are x_1 , x_2 and x_3 . Find

- (a) Lagrangian of the system
- (b) Equation of Motion
- (c) Normal Modes



Solution:

In this problem we can easily guess the normal modes. They are pictured in Figures 11.15(a)-11.15(c). If you think about it a little while you should realize that what's going on here is that the center of mass of the molecule is not accelerating. In mode (c) the central mass is vibrating 180° out of phase with the two end masses. The ratio of the vibrational amplitudes is such that the center of mass remains at rest.

³See footnote 2.









Figure 11.15 Model of a triatomic molecule and its three normal modes for motion in a single line.

Mode (b) obeys the same condition. The central mass remains at rest while the two equal end masses vibrate 180° out of phase with each other, with equal amplitudes, again fixing the center of mass. Mode (a) depicts overall translation of the center of mass at constant velocity.

We could go on and solve the problem using this guess. However, we will not. We will solve it using the general method introduced in the last example, in which we assume that the normal modes are not known. We will ultimately generate a secular equation that, in this example, will be of third order in ω^2 . (There are three coordinates, hence three normal modes and frequencies in the solution.) It turns out that this particular third-order equation will be very easy to solve. Upon obtaining the frequencies of each normal mode, we will then insert them into any one of the equations relating the amplitudes of the displacement coordinates to each other (the matrix equivalent of the secular equation in ω^2), thus obtaining the normal modes.

The Lagrangian of the system is

$$L = T - V$$

= $\left(\frac{m}{2}\dot{x}_{1}^{2} + \frac{M}{2}\dot{x}_{2}^{2} + \frac{m}{2}\dot{x}_{3}^{2}\right) - \left[\frac{K}{2}(x_{2} - x_{1})^{2} + \frac{K}{2}(x_{3} - x_{2})^{2}\right]$

and Lagrange's three equations of motion read

$$m\ddot{x}_{1} + Kx_{1} - Kx_{2} = 0$$

- Kx₁ + M\vec{x}_{2} + 2Kx_{2} - Kx_{3} = 0
- Kx_{2} + m\vec{x}_{3} + Kx_{3} = 0 (11.33)

If a solution of the form $x_1 = A_1 \cos \omega t$, $x_2 = A_2 \cos \omega t$, $x_3 = A_3 \cos \omega t$ exists, then

$$(-m\omega^{2} + K)A_{1} - KA_{2} = 0$$

-KA_{1} + (-M\omega^{2} + 2K)A_{2} - KA_{3} = 0 (11.34)
-KA_{2} + (-m\omega^{2} + K)A_{3} = 0

The secular equation is, thus,

$$\begin{vmatrix} -m\omega^{2} + K & -K & 0\\ -K & -M\omega^{2} + 2K & -K\\ 0 & -K & -m\omega^{2} + K \end{vmatrix} = 0$$
(11.35)

which, upon expanding the determinant and collecting terms, fortuitously becomes the product of three factors

 $\omega^2(-m\omega^2 + K)(-mM\omega^2 + KM + 2Km) = 0$

Equating each of the three factors to zero gives the three normal frequencies of the system:

$$\omega_a = 0$$
 $\omega_b = \left(\frac{K}{m}\right)^{1/2}$ $\omega_c = \left(\frac{K}{m} + 2\frac{K}{M}\right)^{1/2}$

Let us discuss the modes corresponding to these three roots.

(a) The first mode is no oscillation at all but is *pure translation* of the system as a

CM B3 Swhole. If we set $\omega = 0$ in Equations 11.34, we find that $A_1 = A_2 = A_3$ for this mode.

- (b) Setting $\omega = \omega_b$ in Equations 11.34 gives $A_2 = 0$ and $A_1 = -A_3$. In this mode the center particle is at rest while the two end particles vibrate in opposite directions (antisymmetrically) with the same amplitude.
- (c) Finally, setting $\omega = \omega_c$ in Equations 11.34 we obtain the following relations: $A_1 = A_3$ and $A_2 = -2A_1(m/M) = -2A_3(m/M)$. Thus, in this mode the two end particles vibrate in unison while the center particle vibrates oppositely with a different amplitude. The three modes are illustrated in Figure 11.15.

It is interesting to note that the ratio ω_c/ω_b is independent of the constant *K*, namely,

$$\frac{\omega_c}{\omega_b} = \left(1 + 2\frac{m}{M}\right)^{1/2}$$

In the carbon dioxide molecule the mass ratio m/M is very nearly 16:12 for ordinary CO₂ (C₁₂ and O₁₆ atoms). Thus, the frequency ratio

$$\frac{\omega_c}{\omega_b} = \left(1 + 2 \times \frac{16}{12}\right)^{1/2} = \left(\frac{11}{3}\right)^{1/2} = 1.915$$

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CM B2 IF ellega F. Mechanics & hard $(a) \quad m \cdot o^2 r = G \cdot m \cdot M$ $T = \frac{2\pi}{\omega} = \sqrt{\frac{3\pi}{96}} = \sqrt{\frac{942}{7169}} = \sqrt{\frac{942}{7169}} = \sqrt{\frac{3\pi}{96}} = \sqrt{\frac{942}{5.51 \times 10^{3} \cdot 6.67 \times 10^{-11}}}$ = 50635 = 1.4 hours (6) if T'<T or w'rw the body will go off the sweface and will move along an elliptical position (in the stationary ref. frame)

Alya F. Mechanics 2 Land CM B4 IF m ~ 1) (3in) $mv = mv_1 + 3mv_2 (1)$ (a) elastic collision $\frac{mV^2}{2} = \frac{mV_1^2}{7} + 3m \frac{V_2^2}{7}$ $\mathcal{N}^2 = \mathcal{N}_1^2 + 3\mathcal{N}_2^2$ $from(i) \quad v_1 = v - 3v_2$ $O = -6 V V_2 + 12 V_2^2$ solution V2=0 - no collision, therefore $v_2 = \frac{1}{2}v$ $v_1 = v - \frac{3}{2}v = -\frac{1}{2}v$ (opposite direction) (6) in this case v, = V2 and MU= 4mV, $V_l = \frac{V_l}{V_l}$ eaergy dissipated $\mathcal{O} = \frac{mv^2}{2} - 4m \frac{v_1^2}{2} = \frac{m}{2} \left(v^2 - 2v_1^2\right) = \frac{mv^2}{2} \left(1 - \frac{1}{3}\right)$ = 7/mv2