UNL - Department of Physics and Astronomy

Preliminary Examination - Day 1
Thursday, August 12, 2021

This test covers the topics of *Thermodynamics and Statistical Mechanics* (Topic 1) and *Classical Mechanics* (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

Note: If you do more than two problems in a group, only the first two (in the order they appear in this handout) will be graded. For instance, if you do problems A1, A3, and A4, only A1 and A3 will be graded.

WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY

<u>**Thermodynamics and Statistical Mechanics Group A** - Answer only two Group A questions</u>

A1. At *T*=25 °C, the relation between the volume and the pressure of water is $V = 18.066 - 0.715 \times 10^{-3}P + 0.046 \times 10^{-6}P^2$ (cm^3/mole) where *P* is calculated in atm. If 1 mole of water at *T*=25 °C is compressed from 1 atm to 1000 atm, what's the work done by the environment?

A2. Find the thermal expansion coefficient $\alpha = (\partial V/\partial T)_P/V$ and isothermal compressibility $K_T = -(\partial V/\partial P)_T/V$ for ideal gas.

A3. In a game, you repeatedly roll a standard die with the numbers 1 through 6 on its faces. If you roll a 6, the game is over. If you roll any other number, you may roll again.

a. What is the probability that the game is still not over after *N* rolls?
b. What is the probability that you end the game at the *N*-th roll?
c. What is the average number of rolls a player makes in this game?
Hint: $(d/du)u^a = au^{a-1}$

A4. 500 J of heat being conducted from a reservoir at 500 K to one at 300 K. There is no chnage in the reservoir temperatures

(a) What is the change in entropy of the total system in this process?
(b) How much of the heat could be converted into work in an ideal situation?

<u>**Thermodynamics and Statistical Mechanics Group B** - Answer only two Group B questions</u>

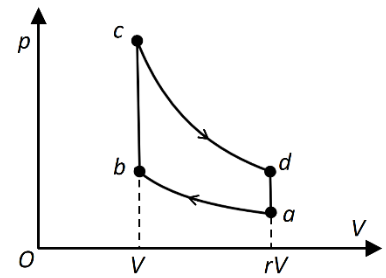
B1. (a) For a diatomic ideal gas near room temperature, what fraction of the heat supplied is available for external work if the gas is expanded (i) at a constant pressure? (ii) At a constant temperature?

(b) Suppose in case (i) the initial temperature of the gas is T_1, and the final is T_2. In case (ii) the initial volume is V_1 and the final is V_2. Calculate the entropy change of one mole of the gas for both cases.

B2. One mole of ideal monoatomic gas undergoes a cyclic process including an isothermal process from state *A* (V_A) to state *B* (with $V_B = 2V_A$), an isobaric process to state *C* (with $V_C = V_A$), and an isochoric process back to state *A*.

(a) Give a sketch of the pV diagram for this cycle;
(b) Calculate the efficiency of the cycle.

B3. The figure shows the pV-diagram for the so-called Otto cycle, which describes gasoline engines. At point *a*, a gas/air mixture has entered the cylinder. This mixture is compressed adiabatically to point *b* and is then ignited. Along the line *b*–*c*, over which the volume is constant and equal to *V*, the burning gasoline adds heat to the system, followed by the adiabatic expansion to point *d* (the "power stroke"). The gas is cooled to the temperature of the outside air along line *d*–*a*, which is at constant volume rV ($r > 1$); during this process, heat is rejected. The gas leaves the engine as exhaust, but an equivalent amount of gasoline and air enters, so we may consider the process to be cyclic.



Consider a one-cylinder Otto-cycle engine with $r = 10.6$. The diameter of the cylinder is 82.5 mm. The distance that the piston moves during the compression is 86.4 mm. The initial pressure (at point *a*) of the gas/air mixture is 8.50×10^4 Pa, and the initial temperature is 300 K (the same as the outside air). Assume that 200 J of heat is added to the cylinder in each cycle by the burning gasoline, and that the gas has $C_V = 20.5\ \mathrm{J/(mol \cdot K)}$ and $\gamma = 1.40$. Assume that the mixture is an ideal gas.

a. Calculate the volume of the air-fuel mixture at point *a* in the cycle.
b. Calculate the amount of the mixture in moles.
c. Calculate the temperature of the mixture at points *b*, *c*, and *d* in the cycle.
d. Calculate the efficiency of this engine and compare it with the efficiency of a Carnot-cycle engine operating between the same maximum and minimum temperature.

B4. An electron in a one-dimensional crystal hops with an equal probabilities to the right or to the left, covering distance *a* at each hop. Suppose the electron has made 10 hops.

(a) What is the probability to find it at the position +2*a* from the original position? At the position +4*a*?
(b) What is the mean number of hops to the right? The mean number of hops to the left?
(c) Suppose the systems is placed in an electric field such that the probability of a hop to the right is 4/5, and to the left 1/5. Answer the same questions as asked in (a) and (b).

Classical Mechanics Group A - Answer only two Group A questions

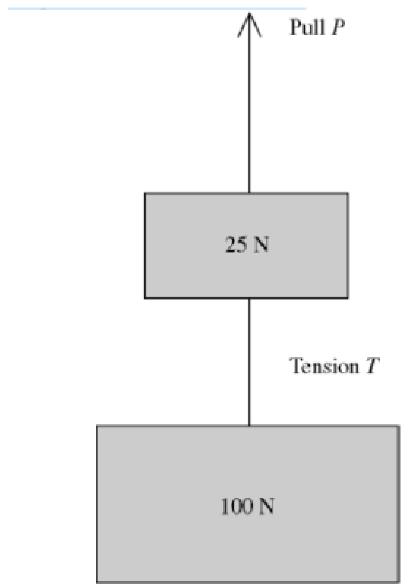


Figure 1: Problem A1.

A1. Two blocks A and B with weight 25 N and 100 N, respectively, are connected by a massless rope and pulled upward by a vertical pull P=150 N. What is the ratio T/P for the scenario shown? What is the acceleration of the blocks?

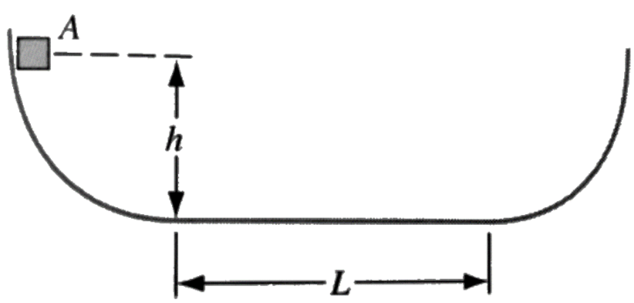


Figure 2: Problem A2

A2. An object of mass m = 234 g slides along a track with elevated ends and a central horizontal part, as shown in Fig. 2. The horizontal part has length L = 2.16 m. The curved portions of the track are frictionless; but in traversing the horizontal part, the object loses 0.688 J of mechanical energy, due to friction. The object is released from rest at point A, a height h = 1.05 m above the horizontal part of the track. Where does the object finally come to rest and in which direction was the object traveling just before it came to rest? What is the coefficient of friction, μ_k, of the horizontal part of the track?

A3. A block of unknown mass is attached to a spring with spring constant 6.5 N/m and undergoes simple harmonic motion with an amplitude of 10 cm. When the block is halfway between its equilibrium position and the endpoint of its motion, its speed is measured to be 30 cm/s. What is the mass of the block? What is the period of motion of the block?

A4. A mass *M* moves horizontally along a smooth rail. A pendulum comprised of a weightless rod of fixed length *l* and a mass *m* is hung from *M*. Determine the Lagrangian for the system.

Classical Mechanics Group B - Answer only two Group B questions

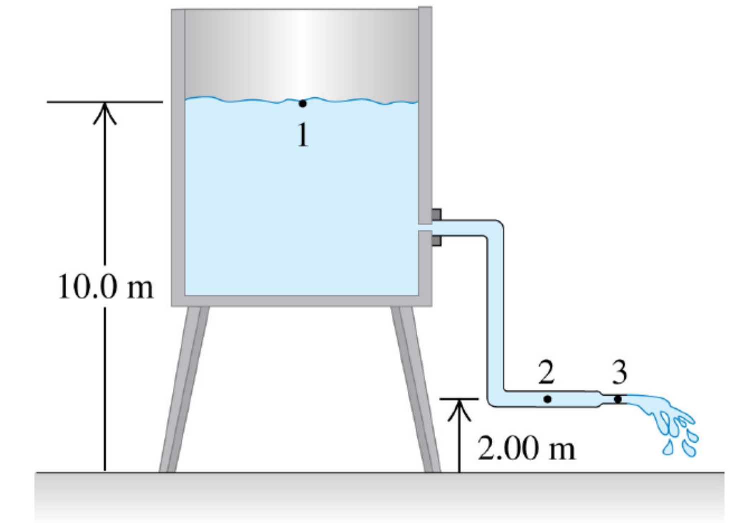


Figure 3: Problem B1

B1. Water flows steadily from an open tank as in Fig. 3. The elevation of point 1 is 10 m, and the elevation of points 2 and 3 is 2 m. The cross-sectional area at point 2 is 0.048 m^2; at point 3 it is 0.016 m^2. The area of the tank is very large compared with the cross-sectional area of the pipe.

(a) What is the velocity of the flow at point 3 and point 2?

(b) At what rate does water flow out in cubic meters per second?

(c) What is the pressure at point 2?

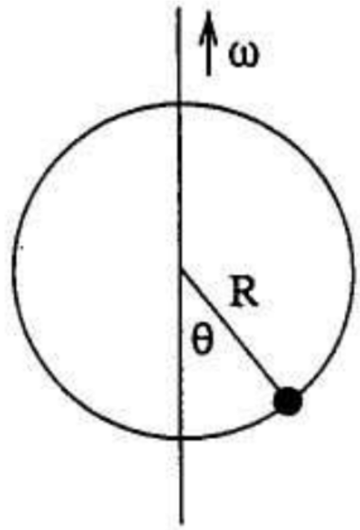


Figure 4: Problem B2

B2. A circular hoop of radius R rotates with angular frequency ω about a vertical axis through the center of the hoop in the plane of the hoop. A bead of mass m slides without friction around the hoop and is subject to gravity.

(a) Determine the Lagrangian and the equations of motion.
(b) Find the equilibrium positions of the bead as functions of ω.
(c) Which of these positions are stable, and how does the stability depend on ω?

B3. An ant crawls outward with a constant speed *v* along a horizontal spoke of a wheel which is rotating with a constant angular velocity ω about the vertical axis.

(a) Find the net force (magnitude and direction) acting on the ant as determined by an observer in the lab frame.
(b) How far can the ant crawl before it starts to slip if the coefficient of static friction between the ant and the spoke is μ?

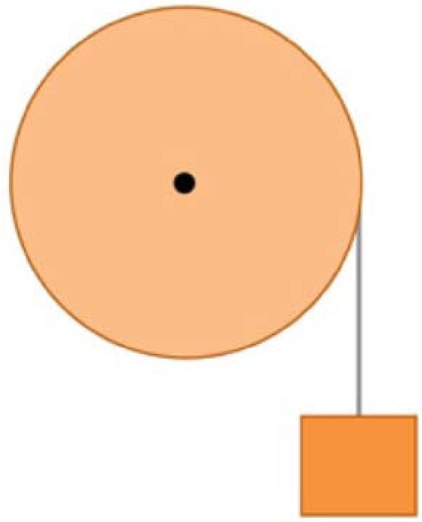


Figure 5: Problem B4.

B4. You need to measure the moment of inertia of a large wheel for rotation about an axis perpendicular to the wheel at its center. You measure the diameter of the wheel to be 0.7 m. Then you mount the wheel on frictionless bearings on a horizontal frictionless axle at the center of the wheel. You wrap a light rope around the wheel and hang an 8.2-kg block of wood from the free end of the rope. You release the system from rest and find that the block descends 12.0 m in 4.0 s. There is no slippage.

(a) What is the moment of inertia of the wheel for this axis?
(b) What is the acceleration of the block?
(c) What is the angular acceleration of the wheel?

Physical Constants

speed of light .... $c = 2.998 \times 10^{8}$ m/s

electron mass .... $m_{\text{el}} = 9.109 \times 10^{-31}$ kg

Boltzmann constant ...... $k_{\text{B}} = 1.381 \times 10^{-23}$ J/K

gravitational constant ..... $G = 6.674 \times 10^{-11}$ $\text{m}^3/(\text{kg}\cdot\text{s}^2)$ g=9.8 m/s^2

Atmospheric pressure.... 101,325 Pa

Avogadro constant .... $N_{\text{A}} = 6.022 \times 10^{23}$ mol^{-1}

gas constant................... $R = 8.314$ J/(mol·K)

Equations That May Be Helpful

TRIGONOMETRY

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

$$\sin\alpha\sin\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$
$$\cos\alpha\cos\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$
$$\sin\alpha\cos\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$
$$\cos\alpha\sin\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$$

For small x:

$$\sin x \approx x - \tfrac{1}{6}x^3$$
$$\cos x \approx 1 - \tfrac{1}{2}x^2$$
$$\tan x \approx x + \tfrac{1}{3}x^3$$

THERMODYNAMICS

Heat capacity = $\quad C_V = N\dfrac{d\langle E\rangle}{dT}$

Clausius' theorem: $\displaystyle\sum_{i=1}^{N}\frac{Q_i}{T_i} \le 0$, which becomes $\displaystyle\sum_{i=1}^{N}\frac{Q_i}{T_i} = 0$ for a reversible cyclic process of N steps.

$$\frac{dp}{dT} = \frac{\lambda}{T\Delta V}$$

Carnot efficiency = 1 – T_c/T_h

Molar heat capacity of diatomic gas is $C_V = \frac{5}{2}R$

For adiabatic processes in an ideal gas with constant heat capacity, $pV^\gamma = \text{const.}$

$$dU = TdS - pdV \qquad dF = -SdT - pdV$$

$$H = U + pV \qquad F = U - TS \qquad G = F + pV \qquad \Omega = F - \mu N$$

$$C_V = \left(\frac{\delta Q}{dT}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V \qquad C_p = \left(\frac{\delta Q}{dT}\right)_p = T\left(\frac{\partial S}{\partial T}\right)_p \qquad TdS = C_V dT + T\left(\frac{\partial S}{\partial V}\right)_T dV$$

$$\kappa = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T \qquad \alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p$$

Binomial distribution

$$W(n_1, n_2) = \frac{N!}{n_1! n_2!} p^{n_1} q^{n_2}, \qquad n_1 + n_2 = N$$

MECHANICS

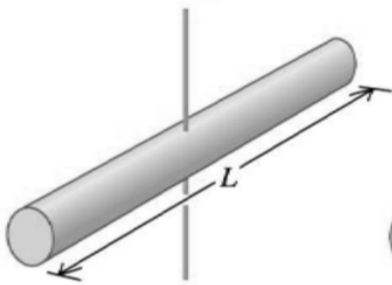
$\mathbf{F}_{cf} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ $\qquad$ $\mathbf{F}_{Cor} = -2m\boldsymbol{\omega} \times \mathbf{v}$

Bernoulli's Equation $P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant}$ where P is pressure and ρ is density

TABLE 9.2 Moments of Inertia of Various Bodies

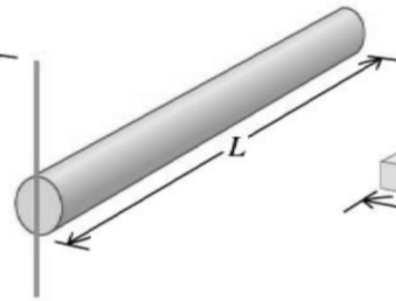
(a) Slender rod, axis through center

$$I = \frac{1}{12}ML^2$$



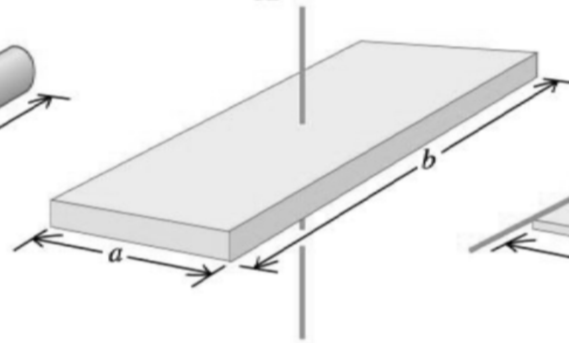
(b) Slender rod, axis through one end

$$I = \frac{1}{3}ML^2$$



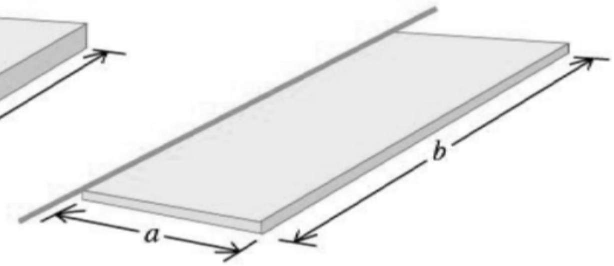
(c) Rectangular plate, axis through center

$$I = \frac{1}{12}M\left(a^2 + b^2\right)$$



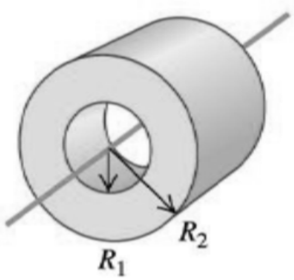
(d) Thin rectangular plate, axis along edge

$$I = \frac{1}{3}Ma^2$$



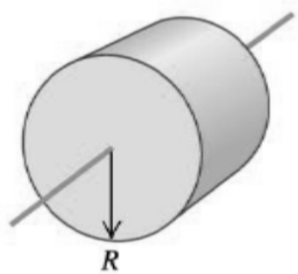
(e) Hollow cylinder

$$I = \frac{1}{2}M\left(R_1^2 + R_2^2\right)$$



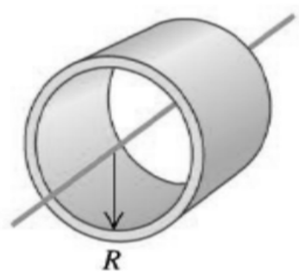
(f) Solid cylinder

$$I = \frac{1}{2}MR^2$$



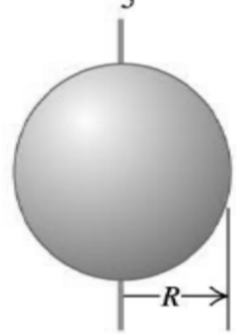
(g) Thin-walled hollow cylinder

$$I = MR^2$$



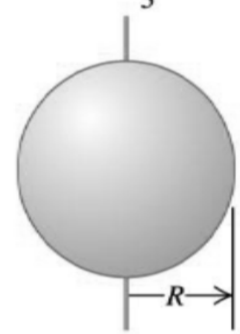
(h) Solid sphere

$$I = \frac{2}{5}MR^2$$



(i) Thin-walled hollow sphere

$$I = \frac{2}{3}MR^2$$



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VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

Curl : $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

CARTESIAN AND SPHERICAL UNIT VECTORS

$$\hat{\mathbf{x}} = (\sin\theta\cos\phi)\hat{\mathbf{r}} + (\cos\theta\cos\phi)\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{y}} = (\sin\theta\sin\phi)\hat{\mathbf{r}} + (\cos\theta\sin\phi)\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

INTEGRALS

$f(x)$	$\int_0^\infty f(x)\,dx$
e^{-ax^2}	$\frac{\sqrt{\pi}}{2\sqrt{a}}$
xe^{-ax^2}	$\frac{1}{2a}$
$x^2e^{-ax^2}$	$\frac{\sqrt{\pi}}{4a^{3/2}}$
$x^3e^{-ax^2}$	$\frac{1}{2a^2}$
$x^4e^{-ax^2}$	$\frac{3\sqrt{\pi}}{8a^{5/2}}$
$x^5e^{-ax^2}$	$\frac{1}{a^3}$
$x^6e^{-ax^2}$	$\frac{15\sqrt{\pi}}{16a^{7/2}}$

$$\int_0^\infty \frac{1}{1+bx^2}dx = \pi/2b^{1/2}$$
$$\int_0^\infty x^n e^{-bx}dx = \frac{n!}{b^{n+1}}$$
$$\int (x^2+b^2)^{-1/2}dx = \ln\left(x+\sqrt{x^2+b^2}\right)$$
$$\int (x^2+b^2)^{-1}dx = \frac{1}{b}\arctan(x/b)$$
$$\int (x^2+b^2)^{-3/2}dx = \frac{x}{b^2\sqrt{x^2+b^2}}$$
$$\int (x^2+b^2)^{-2}dx = \frac{\frac{bx}{x^2+b^2}+\arctan(x/b)}{2b^3}$$
$$\int \frac{x\,dx}{x^2+b^2} = \frac{1}{2}\ln\left(x^2+b^2\right)$$
$$\int \frac{dx}{x(x^2+b^2)} = \frac{1}{2b^2}\ln\left(\frac{x^2}{x^2+b^2}\right)$$
$$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right)$$
$$= -\frac{1}{ab}\operatorname{artanh}\left(\frac{ax}{b}\right)$$