

UNL - Department of Physics and Astronomy

**Preliminary Examination - Day 1**  
**Monday, May 23, 2022**

This test covers the topics of *Thermodynamics and Statistical Mechanics* (Topic 1) and *Classical Mechanics* (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

Note: If you do more than two problems in a group, only the first two (in the order they appear in this handout) will be graded. For instance, if you do problems A1, A3, and A4, only A1 and A3 will be graded.

**WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY**

**Thermodynamics and Statistical Mechanics Group A - Answer only two Group A questions**

**A1.** (a) Find the root-mean-square (rms) speed of molecules in a hydrogen gas at atmospheric pressure. The gas number density is  $2 \times 10^{25} \text{ m}^{-3}$ .

(b) How should the pressure be increased in order to increase the rms speed by a factor of two?

**A2.** Consider an ideal gas in a container of adjustable volume,  $V$ , which in addition allows for control of the temperature,  $T$ . If you want to achieve that the pressure,  $P$ , increases linearly with the volume according to  $P=AV$  with  $A=\text{const}$  you have to increase the temperature while the volume increases.

a) Find the functional form  $T=T(V)$  which allows to realize  $P=AV$ .

b) The temperature of the container is increased to  $T_f$  while the pressure changes according to  $P=AV$  and the volume increases from  $V_i=V_0$  to  $V_f=2V_0$ . Find the work done by the gas in terms of  $n$ ,  $R$  and  $T_f$ . (Hint: In case you could not solve (a) express the work in terms of  $A$  and  $V_0$  for partial credit.)

**A3.** There are 10 coins in a bag. Five of them are normal coins, one coin has two heads and four coins have two tails. You pull one coin out, look at one of its sides and see that it is a tail. What is the probability that it is a normal coin?

**A4.** Air (approximated as a diatomic ideal gas with  $c_v = 2.5R$ ) initially at 293K (20°C) is adiabatically compressed

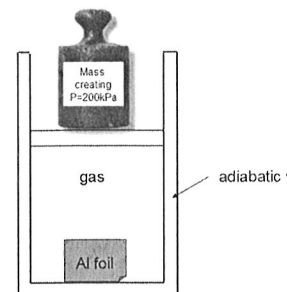
(a) Find the final temperature when the compression ratio  $V_f/V_0$  is 1/10 (as is typical in gasoline engines).

(b) Find the final temperature when the compression ratio  $V_f/V_0$  is 1/20 (as is typical in diesel engines).

**Thermodynamics and Statistical Mechanics Group B - Answer only two Group B questions**

**B1.** Ten grams (10.0 g) of aluminium foil at 30°C (303 K) and 0.50 moles of an ideal gas at 15°C (288 K) are placed in a container whose volume changes to maintain the pressure at 200 kPa. The

foil and the gas come to equilibrium with negligible heat transfer to or from the container. The molar specific heat of the gas is  $c_p = 3.5 R$ , where  $R = 8.314 \text{ J/(mol K)}$  is the universal gas constant and the specific heat capacity of Al is given by  $c_p^M = 0.904 \text{ kJ/(kg K)}$ .



- Find the final temperature of the aluminium foil and the gas.
- Find the work done by the gas

**B2.** A certain volume of water with constant heat capacity  $C_p$  is initially at  $T_i$ . It is brought into contact with a heat reservoir at temperature  $T_r$ .

- What is  $\Delta S_{\text{total}}$ , the entropy change of the entire system (water and reservoir) when the water reaches the temperature of the heat reservoir? Assume that the volume of the water doesn't depend on temperature.  
Express the answer in terms of  $C_p$ ,  $T_i$ , and  $T_r$ .

Hint: Think about sign of the heat flow from or into the reservoir.

- Show that  $\Delta S_{\text{total}}\left(\frac{T_r}{T_i}\right) \geq 0$  for any  $T_r$  and  $T_i$  all by discussing and sketching the function  $\Delta S_{\text{total}}\left(\frac{T_r}{T_i}\right)$ .

**B3.** A cyclic equilibrium process in  $n$  moles of an ideal gas with  $c_v = 2.5 R$  is formed of three sub-processes:

- $a \rightarrow b$  is a constant pressure doubling of the volume;
- $b \rightarrow c$  is at constant volume with decreasing pressure;
- $c \rightarrow a$  is adiabatic.

Assume that  $V_b = V_c = 2V_a$

- Sketch the process on a  $PV$  diagram.
- Find the heat absorbed in part  $a \rightarrow b$  in terms of  $T_a$ ,  $n$ ,  $R$  and a number.
- Find the heat rejected in part  $b \rightarrow c$  in terms of  $T_a$ ,  $n$ ,  $R$  and a number.
- Find the energy efficiency of the cycle. Give a numerical answer.

**B4.** Write down the energy distribution function for molecules in an ideal gas at a temperature  $T$ . Suppose  $T=300$  K.

- (a) Estimate the probability that a molecule has the kinetic energy less than 0.001 eV. Hint: use a suitable approximation in your calculation.
- (b) Estimate the probability that a molecule has the kinetic energy greater than 0.2 eV.

**Classical Mechanics Group A - Answer only two Group A questions**

**A1.** Three identical open-top containers are filled to the brim with water. Toy ducks float in two of the containers, as shown. Rank the containers plus contents according to their weight, e.g. (a) > (b) = (c). Explain your reasoning.

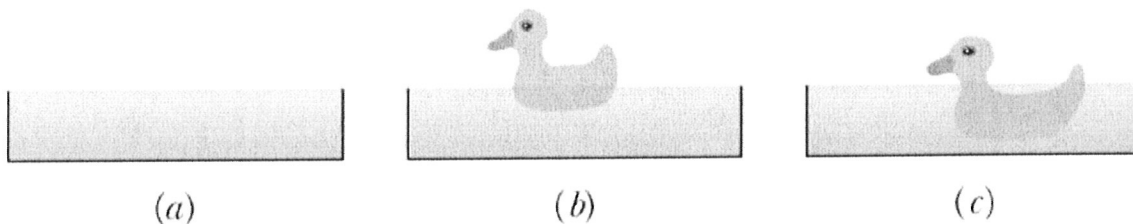


Figure 1: Problem A1

**A2.** A particle slides frictionlessly inside a spherical surface of radius  $R$ , as shown. Show that the motion is simple harmonic for small displacements and find the period of this motion.

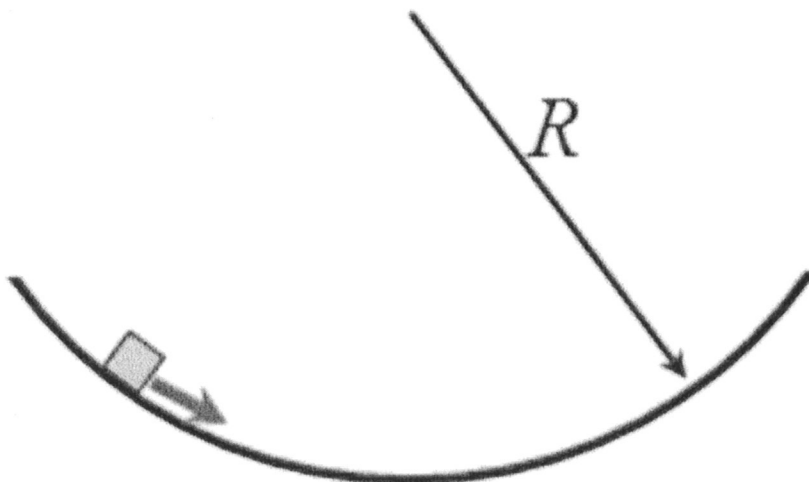


Figure 2: Problem A2

**A3.** Determine the wavelengths of the three lowest-frequency tones produced by a pipe of length  $L$  that is open at both ends.

**A4.** Consider a ball launched from level ground at a fixed angle  $\theta$  with respect to the horizontal and a speed  $v_0$ . The objective is to shoot the ball through a window a distance  $L$  away that is at a height  $h$ . Find an expression for the speed  $v_0$  required to shoot the ball through the window. Your answer should depend only on the parameters defined here and the acceleration due to gravity,  $g$ .

**Classical Mechanics Group B - Answer only two Group B questions**

**B1.** Three pipes with smooth walls rest in an open box (width  $3D$ ) with a horizontal bottom and vertical walls. Two of the pipes have diameter  $D$  and weight  $W_1$  and sit on the bottom of the box with centers separated by distance  $2D$ . The third pipe has diameter  $2D$  and weight  $W_2$  and rests on the other pipes, as shown. Calculate the force on each of the vertical walls.

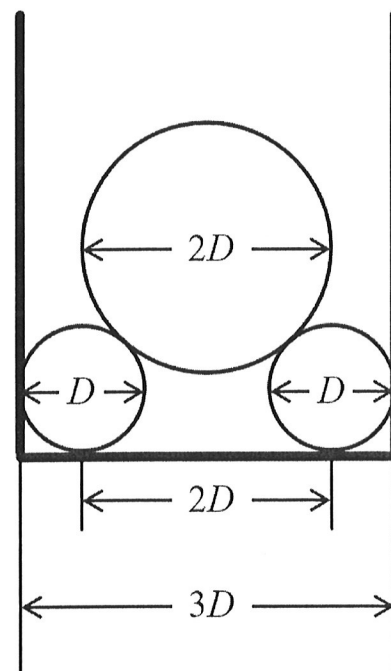


Figure 3: Problem B1

**B2.** A water droplet falling in the atmosphere is spherical. Assume that, as the droplet passes through a cloud, it acquires mass at a rate proportional to  $kA$  where  $k$  is a positive constant and  $A$  its cross-sectional area. Consider a droplet of initial radius  $r_0$  that enters a cloud with velocity  $v_0$ . Assume air resistance force can be neglected. The water mass density is  $\rho$ .

- Show that the radius increases linearly with time  $t$ ;
- Obtain differential equation for the velocity  $v$  as a function of time  $t$  (you don't have to solve it).

**B3.** A uniform density solid cylinder of mass  $M$  and radius  $R$  is free to rotate about its axis of symmetry, which is horizontal. The moment of inertia of the cylinder is  $I = MR^2/2$ . Part of a long cable of negligible mass is wound around the cylinder with the remainder of the cable hanging vertically. A massless spring, with spring constant  $k$ , is attached to the end of the cable, and a block with mass  $m$  is attached to the end of the spring. Determine the Lagrangian using a suitable choice of generalized coordinates and the resulting equations of motion. Find the conjugate momenta and Hamiltonian.

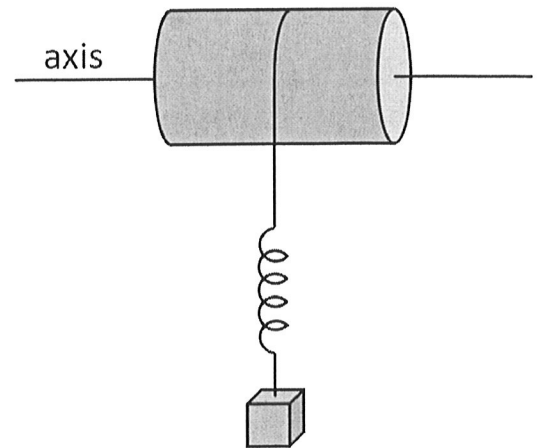


Figure 4: Problem B3

**B4.** A particle of mass  $m$  is moving along a straight line in the presence of a periodic force  $F = F_0 \cos \omega t$ . At  $t=0$  the particle's velocity is  $v_0$ , and its position  $x=0$ .

- (a) Find the particle's position  $x$  as a function of time  $t$ ;
- (b) Find the particle's kinetic energy averaged over many oscillations.

## Physical Constants

speed of light.....  $c = 2.998 \times 10^8$  m/s      Atmospheric pressure.... 101,325 Pa  
 electron mass .....  $m_e = 9.109 \times 10^{-31}$  kg      Avogadro constant .....  $N_A = 6.022 \times 10^{23}$  mol<sup>-1</sup>  
 Boltzmann constant  $k_B = 1.381 \times 10^{-23}$  J/K =  $8.617 \times 10^{-5}$  eV/K      gas constant...  $R = 8.314$  J/(mol · K)  
 Atomic mass unit 1 u =  $1.66 \times 10^{-27}$  kg  
 gravitational constant .....  $G = 6.674 \times 10^{-11}$  m<sup>3</sup> / (kg · s<sup>2</sup>)       $g = 9.8$  m/s<sup>2</sup>

## Equations That May Be Helpful

### TRIGONOMETRY

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$\sin(2\theta) = 2 \sin \theta \cos \theta$$

$$\cos(2\theta) = \cos^2 \theta - \sin^2 \theta = 1 - 2 \sin^2 \theta = 2 \cos^2 \theta - 1$$

$$\sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$

$$\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

$$\cos \alpha \sin \beta = \frac{1}{2} [\sin(\alpha + \beta) - \sin(\alpha - \beta)]$$

For small  $x$  :

$$\sin x \approx x - \frac{1}{6} x^3$$

$$\cos x \approx 1 - \frac{1}{2} x^2$$

$$\tan x \approx x + \frac{1}{3} x^3$$

### THERMODYNAMICS

Heat capacity =  $C_V = N \frac{d\langle E \rangle}{dT}$

Clausius' theorem:  $\sum_{i=1}^N \frac{Q_i}{T_i} \leq 0$ , which becomes  $\sum_{i=1}^N \frac{Q_i}{T_i} = 0$  for a reversible cyclic process of  $N$  steps.

$$\frac{dp}{dT} = \frac{\lambda}{T\Delta V}$$

$$\text{Carnot efficiency} = 1 - T_c/T_h$$

$$\text{Molar heat capacity of diatomic gas is } C_v = \frac{5}{2}R$$

For adiabatic processes in an ideal gas with constant heat capacity,  $pV^\gamma = \text{const.}$

$$\begin{aligned} dU &= TdS - pdV & dF &= -SdT - pdV \\ H &= U + pV & F &= U - TS & G &= F + pV & \Omega &= F - \mu N \end{aligned}$$

$$\begin{aligned} C_v &= \left( \frac{\delta Q}{dT} \right)_v = T \left( \frac{\partial S}{\partial T} \right)_v & C_p &= \left( \frac{\delta Q}{dT} \right)_p = T \left( \frac{\partial S}{\partial T} \right)_p & TdS &= C_v dT + T \left( \frac{\partial S}{\partial V} \right)_T dV \\ \kappa &= -\frac{1}{V} \left( \frac{\partial V}{\partial p} \right)_T & \alpha &= \frac{1}{V} \left( \frac{\partial V}{\partial T} \right)_p \end{aligned}$$

$$\text{Efficiency of a heat engine: } \eta = \frac{W}{|Q_{in}|} = 1 - \frac{|Q_{out}|}{|Q_{in}|}$$

The Maxwell-Boltzmann distribution function

$$f(v)dv = \frac{4}{\pi^{1/2}} \left( \frac{m}{2kT} \right)^{3/2} v^2 e^{-mv^2/kT} dv$$

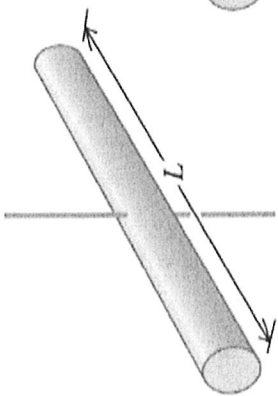
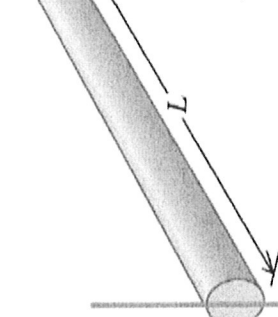
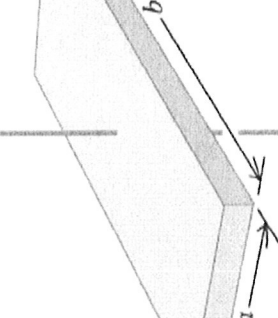
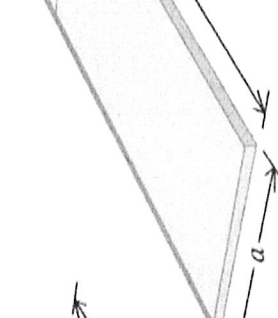
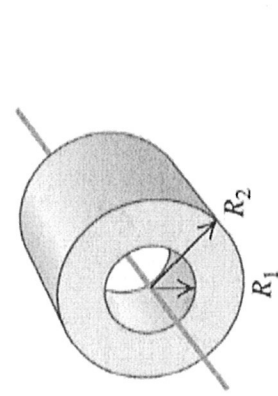
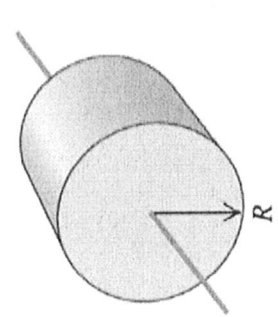
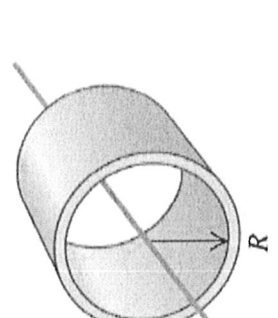
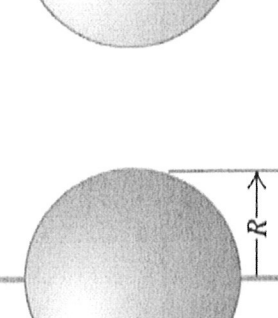
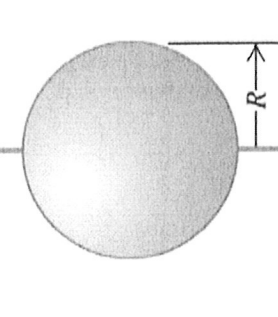
### MECHANICS

$$\mathbf{F}_{cf} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \quad \mathbf{F}_{Cor} = -2m\boldsymbol{\omega} \times \mathbf{v}$$

Bernoulli's Equation  $P + \frac{1}{2}\rho v^2 + \rho gh = \text{constant}$  where  $P$  is pressure and  $\rho$  is density



**TABLE 9.2** Moments of Inertia of Various Bodies

(a) Slender rod, axis through center	$I = \frac{1}{12}ML^2$	(b) Slender rod, axis through one end	$I = \frac{1}{3}ML^2$	(c) Rectangular plate, axis through center	$I = \frac{1}{12}M(a^2 + b^2)$	(d) Thin rectangular plate, axis along edge	$I = \frac{1}{3}Ma^2$
							
(e) Hollow cylinder	$I = \frac{1}{2}M(R_1^2 + R_2^2)$	(f) Solid cylinder	$I = \frac{1}{2}MR^2$	(g) Thin-walled hollow cylinder	$I = MR^2$	(h) Solid sphere	$I = \frac{2}{5}MR^2$
							
						(i) Thin-walled hollow sphere	$I = \frac{2}{3}MR^2$
							

## VECTOR DERIVATIVES

**Cartesian.**  $d\mathbf{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}; \quad d\tau = dx\,dy\,dz$

**Gradient :**  $\nabla f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$

**Divergence :**  $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

**Curl :**  $\nabla \times \mathbf{v} = \left( \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{x} + \left( \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{y} + \left( \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{z}$

**Laplacian :**  $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

**Spherical.**  $d\mathbf{l} = dr\hat{r} + r\,d\theta\hat{\theta} + r\sin\theta\,d\phi\hat{\phi}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

**Gradient :**  $\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}$

**Divergence :**  $\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}v_\phi$

**Curl :**  $\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[ \frac{\partial}{\partial \theta}(\sin\theta v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right] \hat{r}$

$$+ \frac{1}{r}\left[ \frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(rv_\phi) \right] \hat{\theta} + \frac{1}{r}\left[ \frac{\partial}{\partial r}(rv_\theta) - \frac{\partial v_r}{\partial \theta} \right] \hat{\phi}$$

**Laplacian :**  $\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$

**Cylindrical.**  $d\mathbf{l} = ds\hat{s} + s\,d\phi\hat{\phi} + dz\hat{z}; \quad d\tau = s\,ds\,d\phi\,dz$

**Gradient :**  $\nabla f = \frac{\partial f}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial f}{\partial \phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{z}$

**Divergence :**  $\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(sv_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$

**Curl :**  $\nabla \times \mathbf{v} = \left[ \frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right] \hat{s} + \left[ \frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s} \right] \hat{\phi} + \frac{1}{s}\left[ \frac{\partial}{\partial s}(sv_\phi) - \frac{\partial v_s}{\partial \phi} \right] \hat{z}$

**Laplacian :**  $\nabla^2 f = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial f}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$

## VECTOR IDENTITIES

## Triple Products

- (1)  $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$
- (2)  $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

## Product Rules

- (3)  $\nabla(fg) = f(\nabla g) + g(\nabla f)$
- (4)  $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$
- (5)  $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$
- (6)  $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$
- (7)  $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$
- (8)  $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

## Second Derivatives

- (9)  $\nabla \cdot (\nabla \times \mathbf{A}) = 0$
- (10)  $\nabla \times (\nabla f) = 0$
- (11)  $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

## FUNDAMENTAL THEOREMS

**Gradient Theorem :**  $\int_a^b (\nabla f) \cdot d\mathbf{l} = f(b) - f(a)$

**Divergence Theorem :**  $\int (\nabla \cdot \mathbf{A})\,d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

**Curl Theorem :**  $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

**CARTESIAN AND SPHERICAL UNIT VECTORS**

$$\hat{x} = (\sin \theta \cos \phi) \hat{r} + (\cos \theta \cos \phi) \hat{\theta} - \sin \phi \hat{\varphi}$$

$$\hat{y} = (\sin \theta \sin \phi) \hat{r} + (\cos \theta \sin \phi) \hat{\theta} + \cos \phi \hat{\varphi}$$

$$\hat{z} = \cos \theta \hat{r} - \sin \theta \hat{\theta}$$

**INTEGRALS**

$f(x)$	$\int_0^\infty f(x) dx$
$e^{-ax^2}$ .....	$\frac{\sqrt{\pi}}{2\sqrt{a}}$
$xe^{-ax^2}$ .....	$\frac{1}{2a}$
$x^2 e^{-ax^2}$ .....	$\frac{\sqrt{\pi}}{4a^{3/2}}$
$x^3 e^{-ax^2}$ .....	$\frac{1}{2a^2}$
$x^4 e^{-ax^2}$ .....	$\frac{3\sqrt{\pi}}{8a^{5/2}}$
$x^5 e^{-ax^2}$ .....	$\frac{1}{a^3}$
$x^6 e^{-ax^2}$ .....	$\frac{15\sqrt{\pi}}{16a^{7/2}}$

$$\int_0^\infty \frac{1}{1+bx^2} dx = \pi / 2b^{1/2}$$

$$\int_0^\infty x^n e^{-bx} dx = \frac{n!}{b^{n+1}}$$

$$\int (x^2 + b^2)^{-1/2} dx = \ln(x + \sqrt{x^2 + b^2})$$

$$\int (x^2 + b^2)^{-1} dx = \frac{1}{b} \arctan(x/b)$$

$$\int (x^2 + b^2)^{-3/2} dx = \frac{x}{b^2 \sqrt{x^2 + b^2}}$$

$$\int (x^2 + b^2)^{-2} dx = \frac{bx}{x^2 + b^2} + \arctan(x/b) \over 2b^3$$

$$\int \frac{x dx}{x^2 + b^2} = \frac{1}{2} \ln(x^2 + b^2)$$

$$\int \frac{dx}{x(x^2 + b^2)} = \frac{1}{2b^2} \ln\left(\frac{x^2}{x^2 + b^2}\right)$$

$$\int \frac{dx}{a^2 x^2 - b^2} = \frac{1}{2ab} \ln\left(\frac{ax-b}{ax+b}\right) = -\frac{1}{ab} \operatorname{artanh}\left(\frac{ax}{b}\right)$$