UNL - Department of Physics and Astronomy

Preliminary Examination - Day 2
Friday, August 12, 2022

This test covers the topics of *Quantum Mechanics* (Topic 1) and *Electrodynamics* (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

Note: If you do more than two problems in a group, only the first two (in the order they appear in this handout) will be graded. For instance, if you do problems A1, A3, and A4, only A1 and A3 will be graded.

WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY

Quantum Mechanics Group A - Answer only two Group A questions

A1. A muonic atom consists of a proton and negatively-charged muon whose mass is 207 times the mass of electron.

(a) What is the bound state energy for the n=7 excited state and the ground state?
(b) What is the wavelength of a photon emitted from the transition from the 7th excited state to the ground state?

A2. Consider the operator $B = e^A$, where the operator *A* is hermitian. Is B hermitian? Prove your answer.

A3. A spin-2 particle is in the spin state $m_s = 1$. Calculate the angle between its spin vector and the *z* axis.

A4. A free particle of mass *m* is moving in one dimension. At *t*=0 the particle's wavefunction is

$$\psi(x,0)=\exp(ikx)$$

(a) List all quantities (operators) whose eigenstate this wavefunction is, and corresponding eigenvalues;

(b) Write down the particle's wavefunction at time *t*;

(c) answer questions (a) and (b) if $\psi(x,0)=\sin(kx)$.

Quantum Mechanics Group B - Answer only two Group B questions

B1. Consider the angular part of a wave function $\psi(\theta,\phi) = 3\sin\theta\cos\theta e^{i\phi} - 2(1-\cos^2\theta)e^{2i\phi}$.

a. Write ψ in terms of spherical harmonics
b. Is ψ an eigenfunction of L^2? Of L_z? Show work.
c. Find the probability of measuring $2\hbar$ for the z component of the orbital angular momentum.

B2. Consider a physical system whose Hamiltonian *H* and an operator A are given by

$$E = E_0 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad A = a \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$$

,

Where E_0 has the dimension of energy.

a. If we measure the energy, what values can we obtain?
b. Suppose we measure the energy, and find it is 0. Immediately afterwards, we measure *A*. What values of *A* can we obtain and with what probabilities?

B3. An electron at rest is in the eigenstate of S_x corresponding to the maximum eigenvalue. At t=0 a uniform magnetic field directed along *z* axis is switched on.

(a) Find the electron wavefunction at t=0;
(b) Find the electron wavefunction at t>0;
(c) Find the expectation values of S_x, S_y, and S_z as functions of t at t>0.

B4. A free particle of mass *m* at t=0 is described by the following one dimensional momentum-space wavefunction

$$\phi(p) = C \exp(-|p|/p_0)$$

where *C* and p_0 are constants

(a) Find *C* by normalizing $\phi(p)$;

(b) Find the corresponding wavefunction in the position space $\psi(x)$;

(c) Using the definition of Δ p and Δ x as the full-width at half maximum for momentum and position distributions respectively, show that $\Delta x\, \Delta p > \hbar/2$

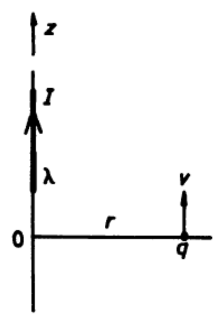
(d)Does this state have a definite parity? Explain.

(e)Write down an integral representation for the wave function $\psi(x,t)$ at t>0 (you don't have to do the integral)

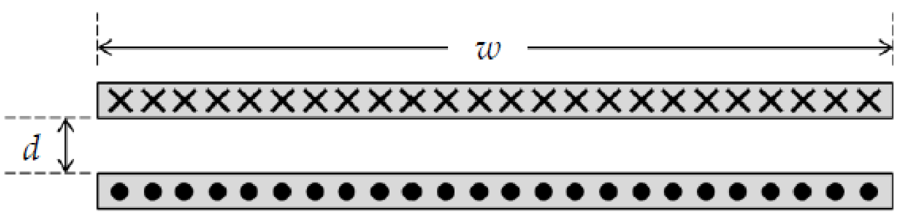
<u>**Electrodynamics Group A** - Answer only two Group A questions</u>

A1. Two uniform infinite sheets of electric charge densities $+\sigma$ and $-\sigma$ intersect at right angles. Find the magnitude and direction of the electric field everywhere and sketch the lines of electric field **E**.

A2. A particle with charge q is traveling with velocity v parallel to a wire with a uniform linear charge distribution λ per unit length. The wire also carries a current I as shown in the figure. What must the velocity be for the particle to travel in a straight line parallel to the wire, a distance r away?

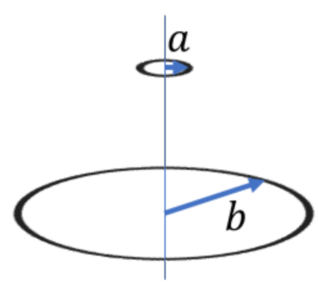


A3. Equal but opposite currents $+I$ and $-I$ flow in two long parallel strips as shown in the figure. The currents are uniformly dis-tributed over the strips. The width of each strip is w, and the distance between them is d ($w \gg d$). The length of the strip is much greater than w.



a. Find the magnetic field between the strips. (Neglect edge effects).
b. Calculate the magnetic field energy per unit length.
c. What is the self-inductance per unit length?

A4. A small circular loop of wire of radius a moves away from a parallel large loop of radius b ($b \gg a$) along the line joining the loop centers so that the distance between their centers varies as $z=z_0+vt$. The velocity is directed perpendicular to the plane of the loop. A constant current I is maintained in the large loop.



(a) Find the current in the small loop I' as a function of time t, if its electrical resistance is R. Neglect the effect of self-inductance;

(b) Show the direction of the current in the small loop;

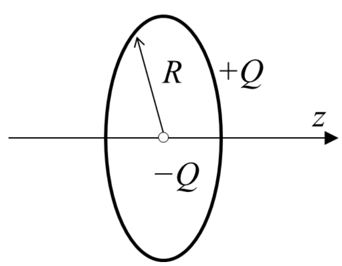
(c) Find the behavior of the current at large time, that is at $vt \gg z_0$ and $vt \gg b$ (You should get an analytical expression, answers like "current goes to 0" will not be accepted).

Electrodynamics Group B - Answer only two Group B questions

B1. A ring of radius R has a total charge $+Q$ uniformly distributed on it.

1) Find electric field **E** at the axis of the ring (the z-axis). From $\mathbf{E}(z)$, find the electrostatic potential at the z-axis from the field $\mathbf{E}(z)$. Suppose the potential is taken to be zero at an infinite point.

2) Consider a point charge $-Q$ at the center of the ring constrained to slide along the z-axis. Show that the charge will execute simple harmonic motion for small displacements perpendicular to the plane of the ring.

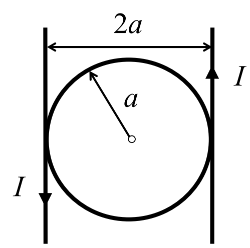


B2. A sphere of radius R_1 has charge density ρ uniform within its volume, except for a small spherical hollow region of radius R_2 located a distance a from the center.

1) Find the electric field E at the center of the hollow sphere.

2) Find the potential Φ at the same point. Suppose the potential is taken to be zero at an infinite point.

B3. Two parallel straight wires of infinite length are separated by distance $2a$ and carry current I in opposite directions, as shown in the figure. A circular conducting ring of radius a lies in the plane of the wires between them. The ring is electrically insulated from the wires. Find the coefficient of mutual inductance between the circular conductor and the two straight wires.



B4. Consider an electromagnetic wave in free space of the form

$$\mathbf{E}(x,y,z,t)=\mathbf{E}_0(x,y)e^{i(kz-\omega t)}, \quad \mathbf{B}(x,y,z,t)=\mathbf{B}_0(x,y)e^{i(kz-\omega t)}.$$

where the amplitudes $\mathbf{E}_0$ and $\mathbf{B}_0$ are in the *xy* plane.

Using Maxwell's equations, find the relation between *k* and *ω* as well as between $\mathbf{E}_0(x,y)$ and $\mathbf{B}_0(x,y)$.

Physical constants

speed of light .... $c=2.998\times10^{8}$ m/s

Planck's constant .... $h=6.626\times10^{-34}$ J·s

Planck's constant / 2π $\hbar=1.055\times10^{-34}$ J·s

Boltzmann constant .. $k_B=1.381\times10^{-23}$ J/K

elementary charge ..... $e=1.602\times10^{-19}$ C

electric permittivity ... $\varepsilon_0=8.854\times10^{-12}$ F/m

magnetic permeability $\mu_0=1.257\times10^{-6}$ H/m

molar gas constant...... $R=8.314$ J / mol·K

Avogadro constant .... $N_A=6.022\times10^{23}$ mol^{-1}

fine structure constant $\alpha=ke^2/(\hbar c)$

electrostatic const.... $k=(4\pi\varepsilon_0)^{-1}=8.988\times10^{9}$ m/F

electron mass .... $m_{el}=9.109\times10^{-31}$ kg

electron rest energy 511.0 keV

Compton wavelength $\lambda_C=h/m_{el}c=2.426$ pm

proton mass .... $m_p=1.673\times10^{-27}$ kg $=1836m_{el}$

1 bohr .... $a_0=\hbar^2/ke^2m_{el}=0.5292$ Å

1 hartree (= 2 Ry) $E_h=\hbar^2/m_{el}a_0^{\,2}=27.21$ eV

gravitational constant $G=6.674\times10^{-11}$ m^3/ kg s^2

hc .... $hc=1240$ eV·nm

1 Ry =13.6 eV

Equations That May Be Helpful

TRIGONOMETRY

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$
$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$
$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

$$\sin(2\theta)=2\sin\theta\cos\theta$$
$$\cos(2\theta)=\cos^2\theta-\sin^2\theta=1-2\sin^2\theta=2\cos^2\theta-1$$

$$\sin\alpha\sin\beta=\tfrac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$$
$$\cos\alpha\cos\beta=\tfrac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$$
$$\sin\alpha\cos\beta=\tfrac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$
$$\cos\alpha\sin\beta=\tfrac{1}{2}\left[\sin(\alpha+\beta)-\sin(\alpha-\beta)\right]$$

$\cos(ix)$=$\cosh(x)$
$\sin(ix)$=$i\sinh(x)$

For small x:
$$\sin x\approx x-\tfrac{1}{6}x^3$$
$$\cos x\approx 1-\tfrac{1}{2}x^2$$
$$\tan x\approx x+\tfrac{1}{3}x^3$$

QUANTUM MECHANICS

$$\left[AB,C\right]=A\left[B,C\right]+\left[A,C\right]B$$

Angular momentum: $[L_x,L_y]=i\hbar L_z \quad et\ cycl.$

Ladder operators:
$$L_+\,|\,\ell,m\rangle=\hbar\sqrt{(\ell+m+1)(\ell-m)}\,|\,\ell,m+1\rangle$$
$$L_-\,|\,\ell,m\rangle=\hbar\sqrt{(\ell+m)(\ell-m+1)}\,|\,\ell,m-1\rangle$$

Gyromagneic ratio for electron = e/m

Pauli matrices: $\sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\sigma_y=\begin{pmatrix}0&-i\\i&0\end{pmatrix}$, $\sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

Table Spherical harmonics and their expressions in Cartesian coordinates.

$Y_{lm}(\theta, \varphi)$	$Y_{lm}(x, y, z)$
$Y_{00}(\theta, \varphi) = \frac{1}{\sqrt{4\pi}}$	$Y_{00}(x, y, z) = \frac{1}{\sqrt{4\pi}}$
$Y_{10}(\theta, \varphi) = \sqrt{\frac{3}{4\pi}} \cos\theta$	$Y_{10}(x, y, z) = \sqrt{\frac{3}{4\pi}} \frac{z}{r}$
$Y_{1,\pm 1}(\theta, \varphi) = \mp\sqrt{\frac{3}{8\pi}} e^{\pm i\varphi} \sin\theta$	$Y_{1,\pm 1}(x, y, z) = \mp\sqrt{\frac{3}{8\pi}} \frac{x \pm iy}{r}$
$Y_{20}(\theta, \varphi) = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1)$	$Y_{20}(x, y, z) = \sqrt{\frac{5}{16\pi}} \frac{3z^2 - r^2}{r^2}$
$Y_{2,\pm 1}(\theta, \varphi) = \mp\sqrt{\frac{15}{8\pi}} e^{\pm i\varphi} \sin\theta\cos\theta$	$Y_{2,\pm 1}(x, y, z) = \mp\sqrt{\frac{15}{8\pi}} \frac{(x \pm iy)z}{r^2}$
$Y_{2,\pm 2}(\theta, \varphi) = \sqrt{\frac{15}{32\pi}} e^{\pm 2i\varphi} \sin^2\theta$	$Y_{2,\pm 2}(x, y, z) = \mp\sqrt{\frac{15}{32\pi}} \frac{x^2 - y^2 \pm 2ixy}{r^2}$

Radial functions for the hydrogen atom $R_{nl}(r)$

$$R_{10}(r) = \frac{2}{a_0^{3/2}} \exp(-r/a_0) \qquad R_{20}(r) = \frac{2}{(2a_0)^{3/2}}[1 - r/(2a_0)] \exp[-r/(2a_0)]$$

$$R_{21}(r) = \frac{r}{24^{1/2} a_0^{5/2}} \exp[-r/(2a_0)]$$

ELECTROSTATICS

$$\oiint_S \mathbf{E}\cdot\hat{\mathbf{n}}\,da = \frac{q_{\text{encl}}}{\varepsilon_0} \qquad \mathbf{E} = -\nabla V \qquad \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{E}\cdot d\boldsymbol{\ell} = V(\mathbf{r}_1) - V(\mathbf{r}_2) \qquad V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

Work done $W = -\int_{\mathbf{a}}^{\mathbf{b}} q\mathbf{E}\cdot d\boldsymbol{\ell} = q\left[V(\mathbf{b}) - V(\mathbf{a})\right]$ Energy stored in elec. field: $W = \frac{1}{2}\varepsilon_0 \int_V E^2 d\tau = Q^2/2C$

Multipole expansion: $\Phi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0 r} + \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{r}\cdot\mathbf{p}}{r^3} + \frac{1}{4\pi\varepsilon_0}\frac{1}{2}\sum_{ij} Q_{ij}\frac{x_i x_j}{r^5} + \dots$, in which

$q = \int \rho(\mathbf{r}) d^3\mathbf{r}$ is the monopole moment

$\mathbf{p} = \int \rho(\mathbf{r})\mathbf{r} d^3\mathbf{r}$ is the dipole moment

$Q_{ij} = \int \rho(\mathbf{r})\left[3r_i r_j - r^2\delta_{ij}\right] d^3\mathbf{r}$ is the quadrupole moment (notation: $r_1 = x,\ r_2 = y,\ r_3 = z$)

Relative permittivity: $\varepsilon_{\mathrm{r}} = 1 + \chi_{\mathrm{e}}$

Bound charges

$\rho_{\mathrm{b}} = -\nabla \cdot \mathbf{P}$
$\sigma_{\mathrm{b}} = \mathbf{P} \cdot \hat{\mathbf{n}}$

Parallel-plate: $C = \varepsilon_0 \dfrac{A}{d}$

Spherical: $C = 4\pi\varepsilon_0 \dfrac{ab}{b-a}$

Cylindrical: $C = 2\pi\varepsilon_0 \dfrac{L}{\ln(b/a)}$ (for a length L)

MAGNETOSTATICS

Relative permeability: $\mu_{\mathrm{r}} = 1 + \chi_{\mathrm{m}}$

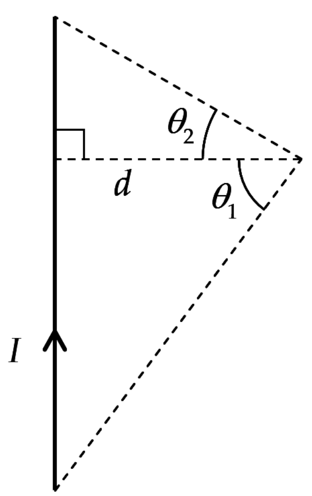
Lorentz Force: $\mathbf{F} = q\mathbf{E} + q(\mathbf{v} \times \mathbf{B})$

Current densities: $I = \int \mathbf{J} \cdot d\mathbf{A}$, $I = \int \mathbf{K} \cdot d\boldsymbol{\ell}$

Biot-Savart Law: $\mathbf{B}(\mathbf{r}) = \dfrac{\mu_0}{4\pi} \displaystyle\int \frac{I d\boldsymbol{\ell} \times \hat{\mathbf{R}}}{R^2}$ ($\mathbf{R}$ is vector from source point to field point $\mathbf{r}$)

Infinitely long solenoid: B-field inside is $B = \mu_0 n I$ (n is number of turns per unit length)

Ampere's law: $\oint \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I_{\mathrm{encl}}$



$$B = \frac{\mu_0 I}{4\pi d}\left(\sin\theta_1 + \sin\theta_2\right)$$

Magnetic field due to a circular loop of radius *a* with current *I* on the symmetry axis at the distance *z* from the loop's center:

$$B = \frac{\mu_0 I a^2}{2(a^2+z^2)^{3/2}}$$

Magnetic dipole moment of a current distribution is given by $\mathbf{m} = I\int d\mathbf{a}$

Force on magnetic dipole: $\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$

Torque on magnetic dipole: $\boldsymbol{\tau} = \mathbf{m}\times\mathbf{B}$

B-field of magnetic dipole: $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{3\hat{\mathbf{r}}(\mathbf{m}\cdot\hat{\mathbf{r}})-\mathbf{m}}{r^3}$

Bound currents

$J_\mathrm{b} = \nabla\times\mathbf{M}$

$K_\mathrm{b} = \mathbf{M}\times\hat{\mathbf{n}}$

Maxwell's Equations in vacuum

1.	$\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}$	Gauss' Law
2.	$\nabla\cdot\mathbf{B} = 0$	no magnetic charge
3.	$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$	Faraday's Law
4.	$\nabla\times\mathbf{B} = \mu_0\mathbf{J} + \varepsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t}$	Ampere's Law with Maxwell's correction

Maxwell's Equations in linear, isotropic, and homogeneous (LIH) media

1.	$\nabla\cdot\mathbf{D} = \rho_\mathrm{f}$	Gauss' Law
2.	$\nabla\cdot\mathbf{B} = 0$	no magnetic charge
3.	$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$	Faraday's Law
4.	$\nabla\times\mathbf{H} = \mathbf{J}_\mathrm{f} + \frac{\partial\mathbf{D}}{\partial t}$	Ampere's Law with Maxwell's correction

Alternative way of writing Faraday's Law: $\oint\mathbf{E}\cdot d\ell = -\frac{d\Phi_B}{dt}$

Mutual and self inductance: $\Phi_2 = M_{21}I_1$, and $M_{21} = M_{12}$; $\Phi = LI$

Energy stored in magnetic field: $W = \frac{1}{2}\mu_0^{-1}\int_V B^2 d\tau = \frac{1}{2}LI^2 = \frac{1}{2}\oint\mathbf{A}\cdot\mathbf{I}\,d\ell$

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl : $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

CARTESIAN AND SPHERICAL UNIT VECTORS

$$\hat{\mathbf{x}} = (\sin\theta\cos\phi)\hat{\mathbf{r}} + (\cos\theta\cos\phi)\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{y}} = (\sin\theta\sin\phi)\hat{\mathbf{r}} + (\cos\theta\sin\phi)\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

INTEGRALS

$$\int x^4 e^{-x} dx = -e^{-x}(x^4 + 4x^3 + 12x^2 + 24x + 24)$$

$$\int_0^\infty x^n e^{-x} dx = n!$$

$f(x)$	$\int_0^\infty f(x)\,dx$
e^{-ax^2}	$\dfrac{\sqrt{\pi}}{2\sqrt{a}}$
xe^{-ax^2}	$\dfrac{1}{2a}$
$x^2 e^{-ax^2}$	$\dfrac{\sqrt{\pi}}{4a^{3/2}}$
$x^3 e^{-ax^2}$	$\dfrac{1}{2a^2}$
$x^4 e^{-ax^2}$	$\dfrac{3\sqrt{\pi}}{8a^{5/2}}$
$x^5 e^{-ax^2}$	$\dfrac{1}{a^3}$
$x^6 e^{-ax^2}$	$\dfrac{15\sqrt{\pi}}{16a^{7/2}}$

$$\int_0^\infty \frac{1}{1+bx^2}dx = \pi / 2b^{1/2}$$

$$\int_0^\infty x^n e^{-bx}dx = \frac{n!}{b^{n+1}}$$

$$\int (x^2+b^2)^{-1/2}dx = \ln\left(x+\sqrt{x^2+b^2}\right)$$

$$\int (x^2+b^2)^{-1}dx = \frac{1}{b}\arctan(x/b)$$

$$\int (x^2+b^2)^{-3/2}dx = \frac{x}{b^2\sqrt{x^2+b^2}}$$

$$\int (x^2+b^2)^{-2}dx = \frac{\dfrac{bx}{x^2+b^2}+\arctan(x/b)}{2b^3}$$

$$\int \frac{x\,dx}{x^2+b^2} = \frac{1}{2}\ln\left(x^2+b^2\right)$$

$$\int \frac{dx}{x(x^2+b^2)} = \frac{1}{2b^2}\ln\left(\frac{x^2}{x^2+b^2}\right)$$

$$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right)$$
$$= -\frac{1}{ab}\operatorname{artanh}\left(\frac{ax}{b}\right)$$