UNL - Department of Physics and Astronomy

# Preliminary Examination - Day 2
# Monday, August 14, 2023

This test covers the topics of *Thermodynamics and Statistical Mechanics* (Topic 1) and *Classical Mechanics* (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

Note: If you do more than two problems in a group, only the first two (in the order they appear in this handout) will be graded. For instance, if you do problems A1, A3, and A4, only A1 and A3 will be graded.

**WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY**

**Thermodynamics and Statistical Mechanics Group A** - Answer only two Group A questions

**A1.** Consider a heat engine which does 2000 BTU of work for every 2500 BTU of heat it absorbs. If the heat sink for the device is a thermal bath of ice water (0°C), what would be the minimum source temperature in °C?

(The BTU, or British Thermal Unit, is a unit of energy equal to 1054 joules.)

**A2**. In a container of negligible mass, 0.140 kg of ice initially at –5°C is added to 0.200 kg of water at temperature 40°C. If no heat lost to the surroundings, what is the final temperature of the system?

**A3.** A monoatomic ideal gas that is initially at pressure $1.00\text{x}10^5$ Pa with a volume 0.100 $m^3$ is compessed adiabatically to a volume 0.0600 $m^3$.

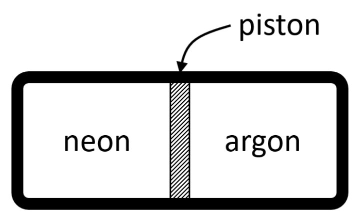
(a) What is the final pressure?
(b) How much work does the gas do?
(c) Does heat flow into or out of the gas? What is the magnitude of the heat flow?
(d) What is the change in internal energy of the gas?

**A4.** 3.00 kg of water at 10°C is mixed with 1.00 kg of water at 80°C. Calculate the entropy change.

**Thermodynamics and Statistical Mechanics Group B** - Answer only two Group B questions

**B1**. A tube leads from a flask in which water is boiling under atmospheric pressure to a calorimeter. The mass of the calorimeter is 0.150 kg, its specific heat capacity is 420 J/kg·K, and it originally contains 0.340 kg of water at $15.0°\text{C}$. Steam is allowed to condense in the calorimeter until its temperature increases to $71.0°\text{C}$, after which the total mass of calorimeter and its content is found to be 0.525 kg. Compute the heat of vaporization of water from these data.

**B2.** A cylindrical container is initially separated by a clamped piston into two compartments of equal volume. The left compartment is filled with one mole of neon gas at a pressure of 4 atmospheres and the right with argon gas at one atmosphere. The gases may be considered as ideal. The whole system is initially at temperature $T$ = 300 K, and is thermally insulated from the outside world. The heat capacity of the cylinder-piston system is $C$ (a constant).




The piston is now unclamped and released to move freely without friction. Eventually, due to slight dissipation, it comes to rest in an equilibrium position. Calculate:

(a) The new temperature of the system (the piston is thermally conductive).
(b) The ratio of final neon to argon volumes.
(c) The total entropy change of the system.
(d) The additional entropy change which would be produced if the piston were removed.
(e) If, in the initial state, the gas in the left compartment were a mole of argon instead of a mole of neon, which, if any, of the answers to (a), (b), (c), and (d) would be different?

**B3.** One mole of real gas that follows the equation of state: $p(V-b)=RT$, where $0<b<V$ is a small constant. Derive the work done by the gas after isothermal expansion from $V$ to $V_f$. Compared with the work done by an ideal gas after isothermal expansion from $V$ to $V_f$, which work is larger?

**B4.** Consider water boiling into water vapor in the atmosphere.

(a) How much is the work done by the water per kg?
(b) What is the change of internal energy per kg?.

*Data you may need: the specific volume (volume per unit mass) is* $V_M^w = 1.0\times10^{-3}\ \mathrm{m^3/kg}$ *and* $V_M^v = 1.8\ \mathrm{m^3/kg}$ *for water and water vapor, respectively. The pressure of the atmosphere is* $1.01\times10^5\ \mathrm{Pa}$*. The latent heat of the liquid-to-vapor transition is 334 kJ/kg.*

**Classical Mechanics Group A** - Answer only two Group A questions

**A1.** A particle of mass $m$ is subjected to two forces: a central force $\mathbf{f}_1 = \frac{f(r)}{r}\mathbf{r}$ and a frictional force $\mathbf{f}_2 = -\lambda \mathbf{v}$ where $\lambda > 0$ is a constant and $\mathbf{v}$ is the velocity of the particle. If the particle initially has angular momentum $\mathbf{J}_0$ about $r = 0$, find its angular momentum for all subsequent times.
*Hint*: Obtain first the differential equation for the angular momentum by taking its time derivative.

**A2.** A particle of mass $m$ is projected from infinity with a velocity $v_0$ in a manner such that it would pass a distance $b$ from a fixed center of an inverse-square repulsive force (magnitude $k/r^2$, where $k$ is a constant) if it were not deflected. Find the distance of the closest approach as a function of $m, b, k,$ and $v_0$.

**A3.** Consider a weight of mass $M_1$ resting at 1/4 of the length of a uniform board mass of $M_2$, supported at each end. What is the ratio of normal reaction forces at each end ?

**A4.** A playground merry-go-round has a diameter 2.4 m and a moment of inertia 2100 kg·m$^2$ about a vertical axis through its center and turns with negligible friction. A child of 38 kg jumps on the edge of the merry-go-round with velocity 1.5 m/s.

(a) If the merry-go-round is initially at rest, what is the angular speed after the child jumped on the merry-go-round ? Treat the child as a point mass.
(b) The child then starts moving towards the center of the merry-go-round with the speed 0.5 m/s. What kind of forces of inertia are acting on the child? Find their directions and magnitudes.

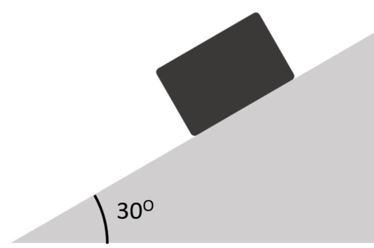
**Classical Mechanics Group B** - Answer only two Group B questions

**B1.** A ladder leans against a wall and the floor. The coefficient of static friction between the ladder and both the wall and the ground is $\mu = 0.4$. What is the critical angle from the vertical such that no slipping occurs?

**B2.** A satellite is in a circular orbit of radius $r_1$ about the center of the Earth. A short thrust of the satellite engine is used to increase the velocity by a factor *s* without changing its direction. This puts the satellite into an elliptical orbit having the perigee (the closest distance) $r_1$ and the apogee (the farthest distance) $r_2$.

(a) Find the satellite's energy per unit mass and the angular momentum per unit mass before and after the thrust. Express them in terms of the gravitational constant *G*, Earth mass *M*, $r_1$ and *s*.
(b) What should be restriction on *s* in order for the orbit to be elliptic?
(c) Find $r_2$ in terms of $r_1$ and *s*.

**B3.** A brick is given an initial speed of 2 m/s up an inclined plane at an angle of 30 degrees from the horizontal (see adjacent figure). The coefficient of (sliding or static) friction is $\mu = 1/(5\sqrt{3})$. After 0.5 second, how far is the brick from its original position? You may take $g = 10 \text{ m/s}^2$.




**B4**. The equation of motion of a point electric charge *e*, of mass *m*, in the field of a magnetic monopole of strength *g* at the origin is:

$$m\ddot{\mathbf{r}} = -ge\frac{\dot{\mathbf{r}} \times \mathbf{r}}{r^3},$$

where the monopole may be taken as infinitely heavy.

a) Show that the kinetic energy is a constant of the motion.
b) Show that $\mathbf{J} = \mathbf{L} + eg\mathbf{r}/r$ is also a constant of the motion, where $\mathbf{L} = m\mathbf{r} \times \dot{\mathbf{r}}$.

## Physical Constants

speed of light................. $c = 2.998\times 10^{8}$ m/s    Atmospheric pressure…. 101,325 Pa

electron mass ................. $m_{\rm el} = 9.109\times 10^{-31}$ kg    Avogadro constant ....... $N_{\rm A} = 6.022\times 10^{23}$ mol$^{-1}$

Boltzmann constant $k_{\rm B} = 1.381\times 10^{-23}$ J/K =8.617x10$^{-5}$ eV/K    gas constant ... $R = 8.314$ J/(mol·K)

Atomic mass unit 1 u=1.66x10$^{-27}$ kg

gravitational constant ..... $G = 6.674\times 10^{-11}$ m$^3$/(kg·s$^2$)    $g$ = 9.8 m/s$^2$

## Equations That May Be Helpful

**TRIGONOMETRY**

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

$$\sin\alpha\sin\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$
$$\cos\alpha\cos\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$
$$\sin\alpha\cos\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$
$$\cos\alpha\sin\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$$

For small $x$:

$$\sin x \approx x - \tfrac{1}{6}x^3$$
$$\cos x \approx 1 - \tfrac{1}{2}x^2$$
$$\tan x \approx x + \tfrac{1}{3}x^3$$

**THERMODYNAMICS**

Specific heat of water: 4.186 kJ/(kg K)
Latent heat of fusion for water 333.5 kJ/kg

Specific heat of water ice: 2.000 kJ/(kg·K)

Heat capacity = $C_V = N\dfrac{d\langle E\rangle}{dT}$

Clausius' theorem: $\sum_{i=1}^{N} \frac{Q_i}{T_i} \leq 0$, which becomes $\sum_{i=1}^{N} \frac{Q_i}{T_i} = 0$ for a reversible cyclic process of $N$ steps.

$$\frac{dp}{dT} = \frac{\lambda}{T\Delta V}$$

Molar heat capacity of a diatomic gas is $C_V = \frac{5}{2}R$

For adiabatic processes in an ideal gas with constant heat capacity, $pV^{\gamma} = \text{const.}$

$$dU = TdS - pdV \qquad dF = -SdT - pdV$$

$$H = U + pV \qquad F = U - TS \qquad G = F + pV \qquad \Omega = F - \mu N$$

$$C_V = \left(\frac{\delta Q}{dT}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V \qquad C_p = \left(\frac{\delta Q}{dT}\right)_p = T\left(\frac{\partial S}{\partial T}\right)_p \qquad TdS = C_V dT + T\left(\frac{\partial S}{\partial V}\right)_T dV$$

$$\kappa = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T \qquad \alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p$$

Efficiency of a heat engine: $\eta = \frac{W}{|Q_{in}|} = 1 - \frac{|Q_{out}|}{|Q_{in}|}$ Carnot efficiency = $1 - T_c/T_h$

The Maxwell-Boltzmann distribution function

$$f(v)dv = \frac{4}{\pi^{1/2}}\left(\frac{m}{2kT}\right)^{3/2} v^2 e^{-mv^2/kT} dv$$

## TABLE 9.2 Moments of Inertia of Various Bodies

(a) Slender rod, axis through center

$$I = \frac{1}{12}ML^2$$

(b) Slender rod, axis through one end

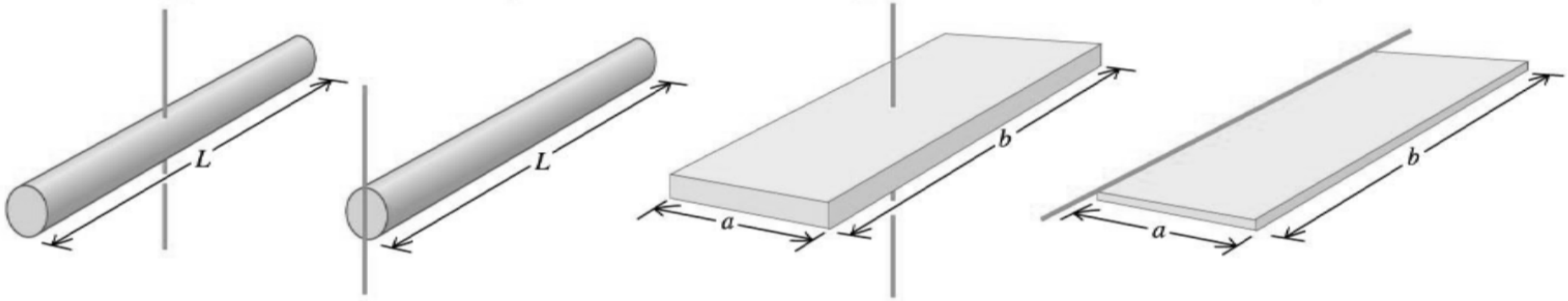
$$I = \frac{1}{3}ML^2$$

(c) Rectangular plate, axis through center

$$I = \frac{1}{12}M(a^2 + b^2)$$

(d) Thin rectangular plate, axis along edge

$$I = \frac{1}{3}Ma^2$$




(e) Hollow cylinder

$$I = \frac{1}{2}M(R_1^2 + R_2^2)$$

(f) Solid cylinder

$$I = \frac{1}{2}MR^2$$

(g) Thin-walled hollow cylinder

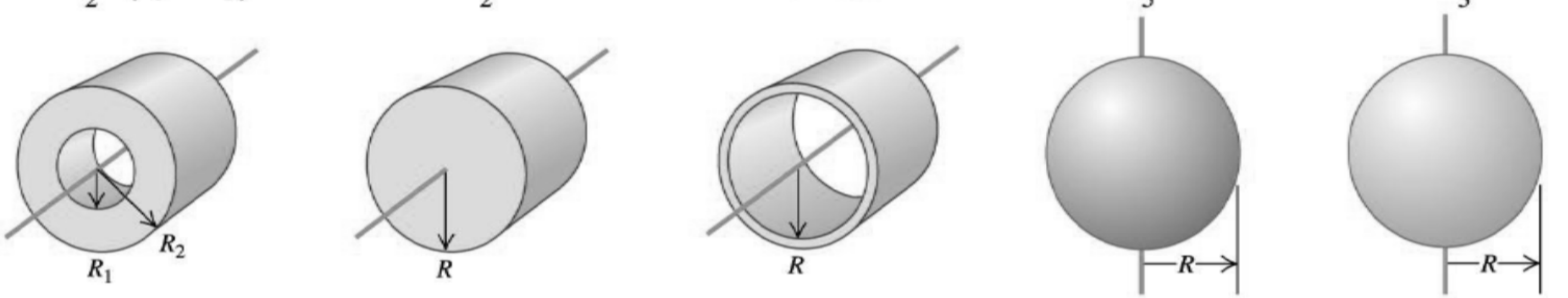
$$I = MR^2$$

(h) Solid sphere

$$I = \frac{2}{5}MR^2$$

(i) Thin-walled hollow sphere

$$I = \frac{2}{3}MR^2$$



## VECTOR DERIVATIVES

**Cartesian.** $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

*Gradient* : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

*Divergence* : $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

*Curl* : $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

*Laplacian* : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

**Spherical.** $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

*Gradient* : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

*Divergence* : $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

*Curl* : $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

*Laplacian* : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$$

**Cylindrical.** $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

*Gradient* : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

*Divergence* : $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$$

*Curl* : $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$$

*Laplacian* : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$$

## VECTOR IDENTITIES

**Triple Products**

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

**Product Rules**

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

**Second Derivatives**

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

## FUNDAMENTAL THEOREMS

**Gradient Theorem :** $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

**Divergence Theorem :** $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

**Curl Theorem :** $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

## CARTESIAN AND SPHERICAL UNIT VECTORS

$$\hat{\mathbf{x}} = (\sin\theta\cos\phi)\hat{\mathbf{r}} + (\cos\theta\cos\phi)\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{y}} = (\sin\theta\sin\phi)\hat{\mathbf{r}} + (\cos\theta\sin\phi)\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

## INTEGRALS

| $f(x)$ | $\int_0^\infty f(x)\,dx$ |
|---|---|
| $e^{-ax^2}$ .............. | $\frac{\sqrt{\pi}}{2\sqrt{a}}$ |
| $xe^{-ax^2}$ .............. | $\frac{1}{2a}$ |
| $x^2 e^{-ax^2}$ .............. | $\frac{\sqrt{\pi}}{4a^{3/2}}$ |
| $x^3 e^{-ax^2}$ .............. | $\frac{1}{2a^2}$ |
| $x^4 e^{-ax^2}$ .............. | $\frac{3\sqrt{\pi}}{8a^{5/2}}$ |
| $x^5 e^{-ax^2}$ .............. | $\frac{1}{a^3}$ |
| $x^6 e^{-ax^2}$ .............. | $\frac{15\sqrt{\pi}}{16a^{7/2}}$ |

$$\int_0^\infty \frac{1}{1+bx^2}dx = \pi / 2b^{1/2}$$
$$\int_0^\infty x^n e^{-bx}dx = \frac{n!}{b^{n+1}}$$
$$\int (x^2+b^2)^{-1/2}dx = \ln\left(x+\sqrt{x^2+b^2}\right)$$
$$\int (x^2+b^2)^{-1}dx = \frac{1}{b}\arctan(x/b)$$
$$\int (x^2+b^2)^{-3/2}dx = \frac{x}{b^2\sqrt{x^2+b^2}}$$
$$\int (x^2+b^2)^{-2}dx = \frac{\frac{bx}{x^2+b^2}+\arctan(x/b)}{2b^3}$$
$$\int \frac{x\,dx}{x^2+b^2} = \frac{1}{2}\ln\left(x^2+b^2\right)$$
$$\int \frac{dx}{x(x^2+b^2)} = \frac{1}{2b^2}\ln\left(\frac{x^2}{x^2+b^2}\right)$$
$$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right) = -\frac{1}{ab}\operatorname{artanh}\left(\frac{ax}{b}\right)$$