A1.

8 points

The reduced mass is given by

$$\mu = \frac{m_\mu m_p}{m_\mu + m_p} = \frac{207 m_e m_p}{207 m_e + m_p} = \frac{207(9.11 \, x \, 10^{-31} kg)(1.67 \, x \, 10^{-27} kg)}{207(9.11 \, x \, 10^{-31} kg) + 1.67 \, x \, 10^{-27} kg}$$

$$= 1.69 \, x \, 10^{-28} kg = 186 m_e$$

Using Under Physical Constants

They can get this from 1 Ryd $= \frac{1}{2}\frac{\hbar^2}{m_e a_0^2} = \frac{\hbar^2}{m_e}\frac{k^2 e^4 m_e^2}{\hbar^4} = \frac{1}{2}\frac{1}{(4\pi\epsilon_0)^2}\frac{e^4 m_e}{\hbar^2}$

9 points

$$E_n = -\frac{e^4}{2(4\pi\varepsilon_0)^2}\frac{\mu}{\hbar^2}\frac{1}{n^2} = -\frac{e^4}{2(4\pi\varepsilon_0)^2}\frac{186 m_e}{\hbar^2}\frac{1}{n^2} = -(186)\frac{13.6 eV}{n^2} = -\frac{2529.6 \, eV}{n^2}$$

If they solve this using mass of muon, give full points for this and

Therefore,

If they dont use reduced mass, $E_n = -(207)\frac{13.6}{n^2}$

$$E_7 = -\frac{2529.6 \, eV}{7^2} = -51.6 \, eV$$

$$E_1 = -\frac{2529.6 \, eV}{1^2} = -2529.6 \, eV$$

$$\frac{hc}{\lambda} = E_7 - E_1$$

8 points

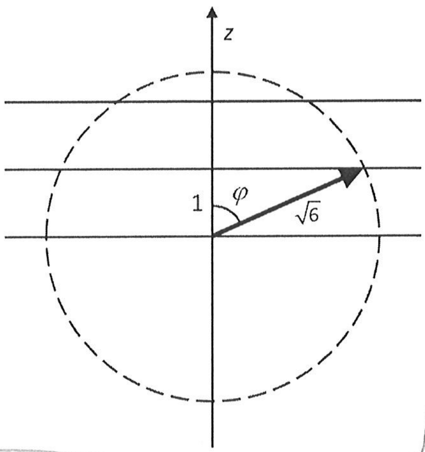
Solving for lambda,

$$\lambda = \frac{hc}{E_7 - E_1} = \frac{1240 \, eV \cdot nm}{-2529.6 \, eV\left(\frac{1}{7^2} - 1\right)} = 0.500 \, nm$$

A2. We consider the operator $B = e^A$, where the operator A is hermitian.

Is B hermitian?

SOLUTION



We have $e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots$ 8 pts

Now when A is hermitian, A^n is hermitian for all (integer) powers n, because for operators $\dots, R, S, T, U \dots$ we have $(\dots RSTU \dots)^\dagger = \dots U^\dagger T^\dagger S^\dagger R^\dagger \dots$, don't need this.

so when $\dots = R = S = T = U = \dots = A = A^\dagger$, we have $(\dots AAAA \dots)^\dagger = \dots A^\dagger A^\dagger A^\dagger A^\dagger \dots = \dots AAAA \dots$ or $(A^n)^\dagger = A^n$ 4 pts

So, B is hermitian, because

$$\underbrace{B^\dagger = (e^A)^\dagger = \left(I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots \right)^\dagger}_{\text{4pts}} = \underbrace{I^\dagger + A^\dagger + \frac{1}{2!}(A^2)^\dagger + \frac{1}{3!}(A^3)^\dagger + \dots}_{\text{4 pts}}$$

$$\underbrace{= I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots = e^A = B}_{\text{5 points}}$$

A3. A spin-2 particle is in the spin state $m_s = 1$. Calculate the angle between its spin vector and the z axis.

SOLUTION

For $\underbrace{s = 2\text{, we have } S^2 = s(s+1)\hbar^2\text{, so the length of } S \text{ is } \sqrt{2 \times 3}\,\hbar = \sqrt{6}\,\hbar}_{\text{8 points}}$. We have $\frac{S}{\hbar} = \sqrt{6}$ and $\underbrace{\frac{m_s}{\hbar} = 1}_{\text{9 points}}$.

$$\underbrace{\text{See diagram: } \varphi = \arccos\left(\frac{m_s}{S}\right) = \arccos\left(\frac{1}{\sqrt{6}}\right) = 66^\circ.}_{\text{8 points}}$$

See Over → A 4

A4 Free particle of mass m moving in 1 dim

$$\Psi(x,0) = e^{ikx}$$

9 points.

a) List all operators whose eigenstate this is.

Momentum $\frac{\hbar}{i}\frac{d}{dx}e^{ikx} = \frac{\hbar}{\cancel{i}}\cancel{i}k\,e^{ikx}$ Eigenvalue of momentum is $\hbar k$.

Kinetic Energy $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}e^{ikx} = -\frac{\hbar^2}{2m}(ik)^2 e^{ikx}$

$$= \frac{\hbar^2}{2m}(-1k)^2 e^{ikx}$$

Energy eigenvalue $= \frac{\hbar^2 k^2}{2m}$

7pts

b) Particle wave function at time t, in terms of k.

$$\Psi(x,t) = e^{ikx}e^{-i\frac{E}{\hbar}t} = e^{ikx}e^{-i\frac{\hbar k^2}{2m}t}$$

$$= e^{i\left(kx - \frac{\hbar k^2}{2m}t\right)}$$

8pts.

c) If $\Psi(x,0) = \sin kx$.

2pts Momentum. $\frac{\hbar}{i}\frac{d}{dx}\sin kx = \frac{\hbar}{i}k\cos kx$ is NOT an eigen function of momentum.

3 pts $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\sin kx = -\frac{\hbar^2}{2m}(-k^2)\sin kx$

Eigen function of energy w/ eigen value $\frac{\hbar^2 k^2}{2m}$

3pts Parity operator $x \to -x$.

$$P\,\Psi(x,0) = \sin k(-x) = -\sin kx$$

Eigen value of -1.

angular part of the ψ of a central potential

B1. Consider the wave function $\psi(\theta,\phi)=3\sin\theta\cos\theta e^{i\phi}-2(1-\cos^2\theta)e^{2i\phi}$.

a. Write ψ in terms of spherical harmonics
b. Is ψ an eigenfunction of L^2? Of L_z? show work
c. Find the probability of measuring $2\hbar$ for the z component of the orbital angular momentum.

SOLUTION

Part a

Using a table of spherical harmonics, we can do the expansion by inspection:

9 points

3 points

Knowing that $Y_{2,+1}=-\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{i\phi} \Rightarrow -\sqrt{\frac{8\pi}{15}}Y_{2,+1}=\sin\theta\cos\theta e^{i\phi}$ so $3\sin\theta\cos\theta e^{i\phi}=-3\sqrt{\frac{8\pi}{15}}Y_{2,+1}$

The second term in ψ may be rewritten: $-2(1-\cos^2\theta)e^{2i\phi}=-2\sin^2\theta e^{2i\phi}$. Now,

4 points

Knowing that $Y_{2,+2}=\sqrt{\frac{15}{32\pi}}\sin^2\theta e^{2i\phi} \Rightarrow \sqrt{\frac{32\pi}{15}}Y_{2,+2}=\sin^2\theta e^{2i\phi}$ so $-2\sin^2\theta e^{2i\phi}=-2\sqrt{\frac{32\pi}{15}}Y_{2,+2}$

2 points

We conclude that $\psi(\theta,\phi)=3\sin\theta\cos\theta e^{i\phi}-2\sin^2\theta e^{2i\phi}=-3\sqrt{\frac{8\pi}{15}}Y_{2,+1}-2\sqrt{\frac{32\pi}{15}}Y_{2,+2}\overset{\text{def.}}{=}c_1Y_{2,+1}+c_2Y_{2,+2}$

Part b

8 points

4 points

$L^2\psi=L^2\left(c_1Y_{2,+1}+c_2Y_{2,+2}\right)=c_1L^2Y_{2,+1}+c_2L^2Y_{2,+2}$. Both Y's are eigenfunctions of L^2 with the same eigenvalue of $6\hbar^2$, so $L^2\psi=6\hbar^2\psi$, and ψ is an eigenfunction of L^2.

4 points

$L_z\psi=L_z\left(c_1Y_{2,+1}+c_2Y_{2,+2}\right)=c_1L_zY_{2,+1}+c_2L_zY_{2,+2}=c_1(\hbar)Y_{2,+1}+c_2(2\hbar)Y_{2,+2}=\hbar\left(c_1Y_{2,+1}+2c_2Y_{2,+2}\right)$. The two Y's by themselves are eigenfunctions of L_z but with different eigenvalues, so ψ is not an eigenfunction of L_z.

Part c

8 points

Of the two spherical harmonics, only the second has $L_z=2\hbar$. 2 points

Thus, the probability is 6 points

$$P(L_z=\hbar)=\frac{|c_2|^2}{|c_1|^2+|c_2|^2}=\frac{\left|-2\sqrt{\frac{32\pi}{15}}\right|^2}{\left|-3\sqrt{\frac{8\pi}{15}}\right|^2+\left|-2\sqrt{\frac{32\pi}{15}}\right|^2}=\frac{\left|-2\sqrt{32}\right|^2}{\left|-3\sqrt{8}\right|^2+\left|-2\sqrt{32}\right|^2}=\frac{4\times32}{9\times8+4\times32}=0.64=64\%$$

B2:

a. (8 points)

energy eigen~~states~~ values $E = E_0 \lambda$

Solve $\begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = 0$

$(1-\lambda)^2 - 1 = 0 \qquad 1-\lambda = \pm 1$

$\lambda_1 = 0, \; \lambda_2 = 2$ — 4 ponts

Therefore $E_1 = 0, \; E_2 = 2E_0$

Eigenvectors — 4 points

$(1-\lambda)C_1 - C_2 = 0$

$C_2 = (1-\lambda)C_1$

$|\phi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \phi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

b. (9 points)

Eigenstates of $A = a\mu$

$\begin{vmatrix} 1-\mu & 1 \\ 1 & 3-\mu \end{vmatrix} = 0$

$\mu^2 - 4\mu + 2 = 0 \qquad \mu_{1,2} = 2 \mp \sqrt{2}$ — 3 points

eigenvectors: $(1-\mu)C_1 + C_2 = 0 \rightarrow C_2 = (\mu - 1)C_1$

$C_2^{(1)} = (1-\sqrt{2})C_1^{(1)} \qquad C_2^{(2)} = (1+\sqrt{2})C_1^{(2)}$ — 3 points

normalize: $[C_1^{(1)}]^2 (1 + (1-\sqrt{2})^2) = 1 \qquad [C_1^{(2)}]^2 (1 + (1+\sqrt{2})^2) = 1$

$[C_1^{(1)}]^2 = \frac{1}{4 - 2\sqrt{2}} \qquad C_1^{(2)} = \frac{1}{4+2\sqrt{2}}$

$|a_1\rangle = \frac{1}{(4-2\sqrt{2})^{1/2}}\begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix} \qquad |a_2\rangle = \frac{1}{(4+2\sqrt{2})^{1/2}}\begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix}$ — 3 points

(8 points)

probability amplitudes:

$\langle a_1|\phi_1\rangle = \frac{1}{2^{1/2}(4-2\sqrt{2})^{1/2}}(1+1-\sqrt{2}) = \frac{\sqrt{2}-1}{(4-2\sqrt{2})^{1/2}}$ — 3 points

probability $P_1 = |\langle a_1|\phi_1\rangle|^2 = \frac{3-2\sqrt{2}}{4-2\sqrt{2}} = 0.146$ — 1 point

$\langle a_2|\phi_1\rangle = \frac{1}{2^{1/2}(4+2\sqrt{2})^{1/2}}(1+1+\sqrt{2}) = \frac{\sqrt{2}+1}{(4+2\sqrt{2})^{1/2}}$ — 3 points

$P_2 = \frac{3+2\sqrt{2}}{4+2\sqrt{2}} = 0.854$ — 1 point

QM B3

At $t=0$ (a) Find eigenstates of $S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

9 points

eigen~~states~~ values of $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ~~are give~~

follows from the equation

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \quad \lambda = \pm 1$$

eigenstates $\chi_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\chi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ … from memory

for $\lambda = +1$ eigenvalue of S_x is $+\frac{\hbar}{2}$, $\chi = \chi_1$

(b)

8 points

$$H = \frac{e}{m}\bar{S}\cdot\bar{B} = \frac{e}{m} S_z B \text{ because } \bar{B} \parallel \hat{z}$$

eigenstates of this Hamiltonian are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

and corresponding eigenvalues $E_{1/2} = \pm\frac{e\hbar}{2m}B$ $E_1 = \frac{e\hbar}{2m}B$

Therefore for $t>0$

$$\psi(t) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-iE_1 t/\hbar} \\ e^{iE_1 t/\hbar} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} f^* \\ f \end{pmatrix}, \quad f = e^{iE_1 t/\hbar}$$

(c)

8 points

$$\langle S_x \rangle = \frac{\hbar}{2}\left(\frac{1}{2}\right)(f \ f^*)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} f^* \\ f \end{pmatrix} = \frac{\hbar}{4}(f^2 + f^{*2})$$

$$= \frac{\hbar}{4}\left(e^{2iE_1 t/\hbar} + e^{-2iE_1 t/\hbar}\right) = \frac{\hbar}{2}\cos\omega t$$

where $\omega = \frac{e}{m}B$ is the Larmor frequency

$$\langle S_y \rangle = \frac{\hbar}{4}(f \ f^*)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} f^* \\ f \end{pmatrix} = \frac{1}{2}(-if^2 + if^{*2}) = \frac{\hbar}{2}\sin\omega t$$

$$\langle S_z \rangle = \frac{\hbar}{4}(f \ f^*)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} f^* \\ f \end{pmatrix} = 0$$

B4. $\Phi(p) = C e^{(-|p|/p_0)}$

a) Find C

5 points

$\int |\Phi(p)|^2 \, dp = 1$ (2 points)

$$\therefore C^2 \int_{-\infty}^{+\infty} e^{-2|p|/p_0} dp = C^2 \left[\int_{-\infty}^{0} e^{+2p/p_0} dp + \int_{0}^{\infty} e^{-2p/p_0} dp \right]$$

$$= C^2 \left[+\frac{p_0}{2} e^{+2p/p_0} \Big|_{-\infty}^{0} + -\frac{p_0}{2} e^{-2p/p_0} \Big|_{0}^{\infty} \right]$$

$$= C^2 \left[\frac{p_0}{2} \left(1 - 0 - (0 - 1) \right) \right] = C^2 p_0 = 1$$

$$\therefore C = 1/\sqrt{p_0}$$

b) Wave function in position space, $\Psi(x)$.

This is the Fourier transform of k space.

5 points

if they forget $\hbar$ don't take off points

$$\Psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \Phi(p) e^{i \frac{p}{\hbar} x} dp \qquad p = \hbar k$$

$$= \frac{1}{\sqrt{2\pi\hbar \, p_0}} \left[\int_{-\infty}^{0} e^{p/p_0} e^{i \frac{p}{\hbar} x} dp + \int_{0}^{\infty} e^{-p/p_0} e^{i \frac{p}{\hbar} x} dp \right]$$

5 points

$$= \frac{1}{\sqrt{2\pi\hbar p_0}} \left[\frac{1}{\frac{1}{p_0} + \frac{i x}{\hbar}} e^{p\left(\frac{1}{p_0} + \frac{i}{\hbar}x\right)} \Big|_{-\infty}^{0} + \frac{1}{-\frac{1}{p_0} + \frac{i}{\hbar}x} e^{-p\left(\frac{1}{p_0} - \frac{i}{\hbar}x\right)} \Big|_{0}^{\infty} \right.$$

$$= \frac{1}{\sqrt{2\pi\hbar p_0}} \left[\frac{p_0 \hbar}{\hbar + i x p_0} (1 - 0) + \frac{\hbar p_0}{-\hbar + i x p_0} (0 - 1) \right]$$

$$\Psi(x) = \frac{1}{\sqrt{2\pi\hbar p_0}} \left[p_0 \hbar \left[\frac{1}{\hbar + i x p_0} - \frac{1}{-\hbar + i x p_0} \right] \right] = \frac{p_0 \hbar}{\sqrt{2\pi\hbar p_0}} \left[\frac{-\hbar + \cancel{i x p_0} - \hbar - \cancel{i x p_0}}{-\hbar^2 - x^2 p_0^2} \right]$$

$$\Psi(x) = \frac{1}{\sqrt{2\pi\hbar p_0}} \left[\frac{+2\hbar p_0 \hbar}{\hbar^2 + x^2 p_0^2} \right]$$

5 pts

c) FWHM of $\phi(p)$

2 pts

$$\frac{1}{\sqrt{2}} e^{-|p_{1/2}|/p_0} = \frac{1}{2\sqrt{2}} \Rightarrow e^{-|p|/p_0} = 1/2$$

$$\ln 1/2 = -\frac{|p|}{p_0}$$

$$-p_0 \ln 1/2 = -|p|$$

$$\Delta p = 2 p_0 (\ln 1/2)$$

FWHM of $\Psi(x)$

$$\frac{1}{\sqrt{2\pi\hbar p_0}} \frac{2\hbar^2 p_0}{\hbar^2 + x_{1/2}^2 p_0^2} = \frac{\cancel{\hbar^2} \cancel{p_0}}{\sqrt{2\pi\hbar p_0}\, \cancel{\hbar^2}}$$

2 pts

i.e. $2\hbar^2 = \hbar^2 + x_{1/2}^2 p_0^2$

$$\sqrt{\frac{2\hbar^2}{p_0^2}} = \Delta x$$

1 pt

$$\Delta x \Delta p = \sqrt{2} \frac{\hbar}{\cancel{p_0}} \cancel{p_0} \ln 1/2 = \sqrt{2}(\hbar)(0.7) \geq \hbar/2$$

3 points

d) Does this state have a definite parity?

Yes because if $x \longrightarrow -x$ nothing changes - even parity.

4 points.

e) Write an integral representation for $\Psi(x,t)$ at $t>0$

$$\Psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{i\frac{p}{\hbar}x} \frac{1}{\sqrt{p_0}} e^{-|p|/p_0} dp.$$

At $t>0$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar p_0}} \int e^{i\frac{p}{\hbar}x} e^{-|p|/p_0} e^{-i\frac{p^2}{2m}t} dp.$$

$E = \frac{p^2}{2m}$ since its a free particle

A1: Two uniform infinite sheets of electric charge densities $+\sigma$ and $-\sigma$ intersect at right angles. Find the magnitude and direction of the electric field everywhere and sketch the lines of electric field **E**.

Solution: First let us consider the infinite sheet of charge density $+\sigma$. The magnitude of the electric field caused by it at any space point is

$$E = \frac{\sigma}{2\varepsilon_0}.$$

5 points

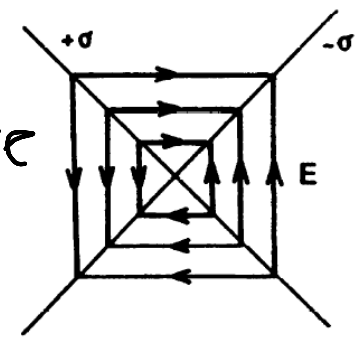
The direction of the electric field is perpendicular to the surface of the sheet. For the two orthogonal sheets of charge densities $\pm\,\sigma$, superposition of their electric fields yields

$$E = \frac{\sqrt{2}\sigma}{2\varepsilon_0}.$$

10 points

The direction of **E** is as shown in Fig. 1.

if wrong sign, remove 4 points.



10 points

Fig. 1

B1: A ring of radius R has a total charge $+Q$ uniformly distributed on it (Fig. 2).

1) Find electric field **E** at the axis of the ring (the z-axis). From $\mathbf{E}(z)$, find the electrostatic potential at the z-axis from the field $\mathbf{E}(z)$.

2) Consider a point charge $-Q$ at the center of the ring constrained to slide along the z-axis. Show that the charge will execute simple harmonic motion for small displacements perpendicular to the plane of the ring.

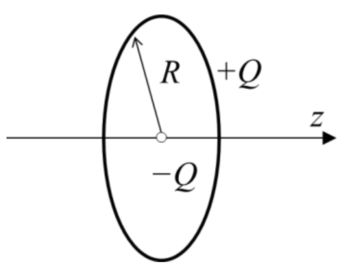


Fig. 2

Solution: 1) The electric field is obtained from Coulomb's law. By symmetry, at the z-axis, the field is pointing along the z direction. A contribution to the z component of the field from a charge dq at any point on the ring is $dE_z = \frac{dq}{4\pi\varepsilon_0 r^2}\cos\theta$, where $r = \sqrt{R^2+z^2}$ and $\cos\theta = \frac{z}{r}$.

Integrating over the charge on the ring, we obtain

$$\mathbf{E}(z) = \frac{Qz}{4\pi\varepsilon_0\left(R^2+z^2\right)^{3/2}}\hat{\mathbf{z}}\,.$$

5 points

if integration error, remove 3points.

The potential is obtained by integrating the field which gives:

$$\Phi(z) = \int_z^\infty E(z')dz' = \frac{Q}{4\pi\varepsilon_0}\int_z^\infty \frac{z'}{\left(R^2+z'^2\right)^{3/2}}dz' = \frac{Q}{4\pi\varepsilon_0\sqrt{R^2+z^2}}\,.$$

10 pints

2) The electrostatic force acting on the change is given by

$$\mathbf{F}(z) = -Q\mathbf{E}(z) = -\frac{Q^2 z}{4\pi\varepsilon_0\left(R^2+z^2\right)^{3/2}}\hat{\mathbf{z}}\,.$$

10 points

For small displacements $z \ll R$, the force F is proportional to z and therefore charge $-Q$ will execute simple harmonic oscillations.

B2. A sphere of radius R_1 has charge density ρ uniform within its volume, except for a small spherical hollow region of radius R_2 located a distance a from the center.

1) Find the electric field E at the center of the hollow sphere.

2) Find the potential Φ at the same point.

Solution: 1) Consider an arbitrary point P of the hollow region (see Fig. 3) and let

$OP = \mathbf{r},\quad O'P = \mathbf{r}',\quad OO' = \mathbf{a},\quad \mathbf{r}' = \mathbf{r} - \mathbf{a}$.

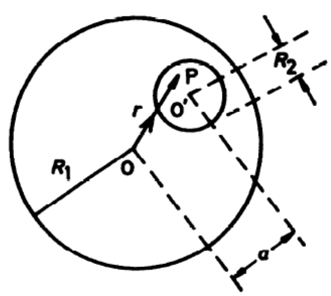


Fig. 3

If there were no hollow region inside the sphere, the electric field at the point P would be

$$\mathbf{E}_1 = \frac{\rho}{3\varepsilon_0}\mathbf{r}\,.$$

If only the spherical hollow region has charge density p the electric field at P would be

$$\mathbf{E}_2 = \frac{\rho}{3\varepsilon_0}\mathbf{r}'\,.$$

2points

Hence the superposition theorem gives the electric field at P as

$$\mathbf{E} = \mathbf{E}_1 - \mathbf{E}_2 = \frac{\rho}{3\varepsilon_0}\mathbf{a}\,.$$

5 points

Thus, the field inside the hollow region is uniform. This of course includes the center of the hollow.

2) Suppose the potential is taken to be zero at an infinite point. Consider an arbitrary sphere of radius R with a uniform charge density ρ. We can find the electric fields inside and outside the sphere as

$$\mathbf{E}(\mathbf{r}) = \begin{cases} \dfrac{\rho}{3\varepsilon_0}\mathbf{r}, & r < R, \\ \dfrac{\rho R^3}{3\varepsilon_0 r^3}\mathbf{r}, & r > R. \end{cases}$$

5 points

Then the potential at an arbitrary point inside the sphere is

$$\Phi = \int_r^R \mathbf{E}\cdot d\mathbf{r} + \int_R^\infty \mathbf{E}\cdot d\mathbf{r} = \frac{\rho}{6\varepsilon_0}\left(3R^2 - r^2\right). \qquad (1)$$

5 points

where r is the distance between this point and the spherical center.

Now consider the problem in hand. If the charges are distributed throughout the sphere of radius R_1, let Φ_1 be the potential at the center O' of the hollow region. If the charge distribution is replaced by a small sphere of uniform charge density ρ of radius R_2 the hollow region, let the potential at O' be Φ_2. Using (1) and the superposition theorem, we obtain

$$\Phi_{O'} = \Phi_1 - \Phi_2 = \frac{\rho}{6\varepsilon_0}\left(3R_1^{\,2} - a^2\right) - \frac{\rho}{6\varepsilon_0}\left(3R_2^{\,2} - 0\right) = \frac{\rho}{6\varepsilon_0}\left[3(R_1^{\,2} - R_3^{\,2}) - a^2\right].$$

8 points

$$V = -\int_a^b \mathbf{E}\cdot d\mathbf{r} = -\int_a^b \frac{Q}{4\pi\varepsilon r^2}\,dr = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{b} - \frac{1}{a}\right).$$

Remove 8 points if integration error

A2. A particle with charge q is traveling with velocity v parallel to a wire with a uniform linear charge distribution λ per unit length. The wire also carries a current I as shown in Fig.5. What must the velocity be for the particle to travel in a straight line parallel to the wire, a distance r away?

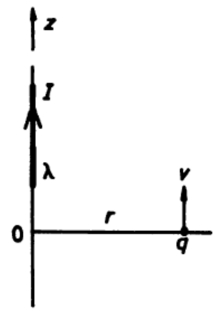


Fig. 5

Solution: Consider a long cylinder of radius r with the axis along the wire. Denote its curved surface for unit length by S and the periphery of its cross section by C. Using Gauss's flux theorem and Ampere's circuital law,

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{\lambda}{\varepsilon_0} \text{ and } \oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I \,,$$

8 points

by the axial symmetry we find

$$\mathbf{E}(\mathbf{r}) = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{\mathbf{r}} \text{ and } \mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\phi}} \,,$$

in cylindrical coordinates $\mathbf{r} = (r, \phi, z)$ with origin at the wire.

in cylindrical coordinates (r, 0, z) with origin 0 at the wire.

The total force acting on the particle which has velocity $\mathbf{v} = v\hat{\mathbf{z}}$ is

$$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} = \frac{q\lambda}{2\pi\varepsilon_0 r}\hat{\mathbf{r}} + \frac{q\mu_0 I}{2\pi r} v(-\hat{\mathbf{r}})$$

12 points

For the particle to maintain the motion along the z direction, this radial force must vanish, i.e.,

$$\frac{q\lambda}{2\pi\varepsilon_0 r} - \frac{q\mu_0 I}{2\pi r} v = 0 \,,$$

resulting in

$$v = \frac{q\lambda}{\mu_0 \varepsilon_0 I} = \frac{\lambda c^2}{I} \,.$$

5 points

B3. Two parallel straight wires of infinite length are separated by distance $2a$ and carry current I in opposite directions, as shown in Fig. 4. A circular conducting ring of radius a lies in the plane of the wires between them. The ring is electrically insulated from the wires. Find the coefficient of mutual inductance between the circular conductor and the two straight wires.

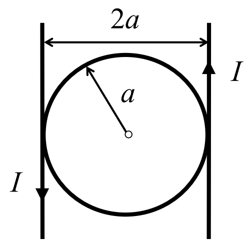
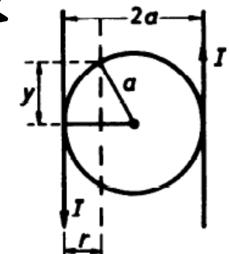


Fig. 4

Solution: The magnetic field at a point between the two conductors at distance r from one conductor is

$$\mathbf{B}(r) = \frac{\mu_0 I}{2\pi}\left(\frac{1}{r} + \frac{1}{2a-r}\right)\hat{\boldsymbol{\phi}}\,.$$

5 points



So, the magnetic flux crossing the area of the ring is given by

$$\phi = \int \mathbf{B}\cdot d\mathbf{s} = 2\int_0^a B(r)\cdot 2y dr = 2\int_0^a \frac{\mu_0 I}{2\pi}\left(\frac{1}{r} + \frac{1}{2a-r}\right)\cdot 2\sqrt{a^2-(a-r)^2}\,dr\,.$$

12 points

Let $x = a - r$ and integrate

$$\phi = 2\int_0^a \frac{\mu_0 I}{\pi}\left(\frac{1}{a-x} + \frac{1}{a+x}\right)\sqrt{a^2-x^2}\,dx = 2\int_0^a \frac{\mu_0 I}{\pi}\left(\frac{2a}{\sqrt{a^2-x^2}}\right)dx = \frac{4\mu_0 Ia}{\pi}\arcsin\frac{x}{a}\bigg|_0^a = 2\mu_0 Ia$$

6 points

The coefficient of mutual inductance is therefore

$$M = \frac{\phi}{I} = 2\mu_0 a\,.$$

2 points if write $M = \frac{\phi}{I}$

B4. Consider an electromagnetic wave in free space of the form

$$\mathbf{E}(x,y,z,t) = \mathbf{E}_0(x,y)e^{i(kz-\omega t)}, \qquad \mathbf{B}(x,y,z,t) = \mathbf{B}_0(x,y)e^{i(kz-\omega t)}.$$

where the amplitudes $\mathbf{E}_0$ and $\mathbf{B}_0$ are in the xy plane.

Using Maxwell's equations, find the relation between k and ω as well as between $\mathbf{E}_0(x,y)$ and $\mathbf{B}_0(x,y)$.

Solution:

1) Calculating the curl of fields **E** and **B**, we find

$$\nabla\times\mathbf{E}=\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ E_x & E_y & E_z\end{vmatrix}=\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & ik\\ E_{0x} & E_{0y} & 0\end{vmatrix}e^{i(kz-\omega t)}=\left[-ikE_{0y}\hat{\mathbf{x}}+ikE_{0x}\hat{\mathbf{y}}+\left(\frac{\partial E_{0y}}{\partial x}-\frac{\partial E_{0x}}{\partial y}\right)\hat{\mathbf{z}}\right]e^{i(kz-\omega t)}.$$

5 points

Similar expression can be found for $\nabla\times\mathbf{B}$.

Maxwell's equations $\nabla\times\mathbf{E}=-\dfrac{\partial\mathbf{B}}{\partial t}$, $\nabla\times\mathbf{B}=\dfrac{1}{c^2}\dfrac{\partial\mathbf{E}}{\partial t}$, can then be written as

$$-ikE_{0y}\hat{\mathbf{x}}+ikE_{0x}\hat{\mathbf{y}}+\left(\frac{\partial E_{0y}}{\partial x}-\frac{\partial E_{0x}}{\partial y}\right)\hat{\mathbf{z}}=\left(-i\omega B_{0x}\hat{\mathbf{x}}-i\omega B_{0y}\hat{\mathbf{y}},\right)\times(-1)$$

$$-ikB_{0y}\hat{\mathbf{x}}+ikB_{0x}\hat{\mathbf{y}}+\left(\frac{\partial B_{0y}}{\partial x}-\frac{\partial B_{0x}}{\partial y}\right)\hat{\mathbf{z}}=-\frac{i\omega}{c^2}E_{0x}\hat{\mathbf{x}}-\frac{i\omega}{c^2}E_{0y}\hat{\mathbf{y}}.$$

10 points

From the z components, we have

$$\frac{\partial E_{0y}}{\partial x}-\frac{\partial E_{0x}}{\partial y}=0,\quad \frac{\partial B_{0y}}{\partial x}-\frac{\partial B_{0x}}{\partial y}=0.$$

4 points

For the xy components we obtain:

$$(-1)\left(E_{0y}\hat{\mathbf{x}}-E_{0x}\hat{\mathbf{y}}\right)=\frac{\omega}{k}B_{0x}\hat{\mathbf{x}}+\frac{\omega}{k}B_{0y}\hat{\mathbf{y}}\ ,\ B_{0y}\hat{\mathbf{x}}-B_{0x}\hat{\mathbf{y}}=\frac{\omega}{kc^2}E_{0x}\hat{\mathbf{x}}+\frac{\omega}{kc^2}E_{0y}\hat{\mathbf{y}}\ ,$$

or, more compactly:

$$\hat{\mathbf{z}}\times\mathbf{E}_0=\frac{\omega}{k}\mathbf{B}_0,\quad \hat{\mathbf{z}}\times\mathbf{B}_0=-\frac{\omega}{kc^2}\mathbf{E}_0.$$

Substituting $\mathbf{E}_0$ from the second equation to the first we have:

$$-\frac{kc^2}{\omega}\hat{\mathbf{z}}\times(\hat{\mathbf{z}}\times\mathbf{B}_0)=-\frac{kc^2}{\omega}\left[\hat{\mathbf{z}}(\hat{\mathbf{z}}\cdot\mathbf{B}_0)-\mathbf{B}_0(\hat{\mathbf{z}}\cdot\hat{\mathbf{z}})\right]=\frac{kc^2}{\omega}\mathbf{B}_0$$

and hence $k=\dfrac{\omega}{c}$.

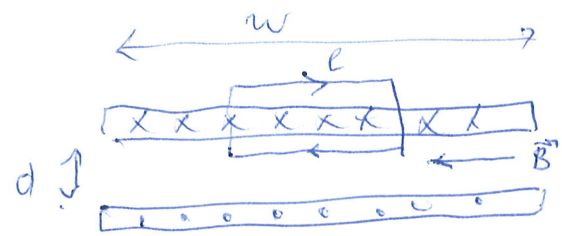
The relationship between $\mathbf{E}_0(x,y)$ and $\mathbf{B}_0(x,y)$ is

$$\hat{\mathbf{z}}\times\mathbf{E}_0=\frac{\omega}{k}\mathbf{B}_0\ \text{ or equivalently }\ \mathbf{E}_0=-\frac{\omega}{k}\hat{\mathbf{z}}\times\mathbf{B}_0.$$

6 points

Algebra mistakes, remove 4 points.

EM A3.



2 points

The surface current density is $K = \frac{I}{w}$

a. According to the Ampere law

$$2 B_1 l = \mu_0 \frac{I}{w} l \, , \quad B_1 = \mu_0 \frac{I}{2w}$$

This is the field due to one strip (between the strips)

The field due to two strips is $B = \mu_0 \frac{I}{w}$

b. $W = \frac{B^2}{2\mu_0} d w S$ where S is the length of the strip

$$\frac{W}{S} = \frac{B^2}{2\mu_0} d w = \frac{d w}{2\mu_0} \left(\frac{\mu_0 I}{w} \right)^2 = \frac{\mu_0 I^2 d}{2w}$$

c. from $W = \frac{L I^2}{2}$

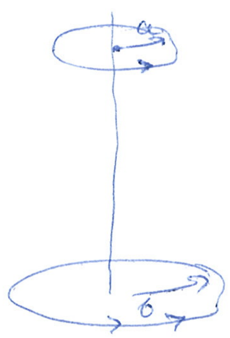
$$\frac{L}{S} = \frac{\mu_0 d}{2w}$$

1. if B is found give 8 points

2. if W is found give 8 points

3 if $\frac{L}{S}$ is found give 7 points

EM A4



The field on the axis due to large loop is

$$B = \frac{\mu_0 I b^2}{2[b^2+(z_0+vt)^2]^{3/2}}$$ } 8 points

since $a \ll b$ we assume that B is uniform in the smaller loop

(a) $\mathcal{E} = -\frac{d}{dt}(B a^2 \pi)$ $\qquad \Phi = B\pi a^2$

8 points

$$I' = -\frac{1}{R}\frac{d\Phi}{dt} = \frac{3}{2R}\frac{\mu_0 I b^2 \pi a^2}{2[b^2+(z_0+vt)^2]^{5/2}}\cdot 2v(z_0+vt)$$

$$= \frac{3}{2}\frac{\pi\mu_0 I b^2 a^2 v(z_0+vt)}{R[b^2+(z_0+vt)^2]^{5/2}}$$ (plus sign)

(b) current in the small loop is in the same direction as in the large loop since the flux is decreasing (Lenz law)

9 points

(c) $vt \gg z_0, b$

$$I' = \frac{3}{2}\frac{\pi\mu_0 I b^2 a^2 v^2 t}{R(vt)^5} = \frac{3}{2}\frac{\pi\mu_0 I b^2 a^2}{R v^3 t^4}$$

EM (B4) alternative solution

$$\vec{E} = \vec{E}_0(x,y)\, e^{i(kz-\omega t)}, \qquad \vec{B} = \vec{B}_0(x,y)\, e^{i(kz-\omega t)}$$

$$\vec{\nabla}\times\vec{E} = \left[-ikE_{oy}\,\hat{x} + ikE_{ox}\,\hat{y} + \left(\frac{\partial E_{oy}}{\partial x} - \frac{\partial E_{ox}}{\partial y}\right)\hat{z}\right] e^{i(kz-\omega t)}$$

$$\frac{\partial \vec{B}}{\partial t} = -i\omega\vec{B} \qquad \frac{\partial \vec{E}}{\partial t} = -i\omega\vec{E}$$

from $\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$

$$-ikE_{oy}\,\hat{x} + ikE_{ox}\,\hat{y} + \left(\frac{\partial E_{oy}}{\partial x} - \frac{\partial E_{ox}}{\partial y}\right)\hat{z} = i\omega B_{ox}\,\hat{x} + i\omega B_{oy}\,\hat{y}$$

from $\vec{\nabla}\times\vec{B} = \dfrac{1}{c^2}\dfrac{\partial\vec{E}}{\partial t}$

$$-ikB_{oy}\,\hat{x} + ikB_{ox}\,\hat{y} + \left(\frac{\partial B_{oy}}{\partial x} - \frac{\partial B_{ox}}{\partial y}\right)\hat{z} = -\frac{i\omega}{c^2}E_{ox}\,\hat{x} - \frac{i\omega}{c^2}E_{oy}\,\hat{y}$$

in components

$$\left.\begin{aligned} -kE_{oy} &= \omega B_{ox}, & kE_{ox} &= \omega B_{oy} \\ -kB_{oy} &= -\frac{\omega}{c^2}E_{ox}, & kB_{ox} &= -\frac{\omega}{c^2}E_{oy} \end{aligned}\right\} (*)$$

combining 1s and 4th eqs, we get

$$\frac{\omega}{k}B_{ox} = \frac{kc^2}{\omega}B_{ox} \quad \text{or} \quad k = \frac{\omega}{c}$$

and the relationship between $\vec{E}_0$ and $\vec{B}_0$ is given by $(*)$,

or $$E_{ox} = \frac{\omega}{k}B_{oy}, \qquad E_{oy} = -\frac{\omega}{k}B_{ox}$$

$$E_{ox} = cB_{oy} \qquad E_{oy} = -cB_{ox}$$