Al.

1. Find the angle $\theta$ at which the rope will settle. Express your answers in terms of the given variables $M, g$, and $a$ as needed.

Solution: Applying Newton's second law to mass $M$,

$$
\begin{aligned}
& \sum F_{y}=0=T \cos \theta-M g \\
& \sum F_{y}=M a=T \sin \theta
\end{aligned}
$$

Solving for $\theta$ yields

$$
\tan \theta=\frac{a}{g}
$$

2. What will the tension $T$ of the rope be once it settles into this angle? Express your answers in terms of the given variables $M, g$, and $a$ as needed.

Solution: Taking the result for $\theta$ obtained in the previous step and substitoting into the equations of motion yields

$$
T=M \sqrt{g^{2}+a^{2}}
$$



AR.


Solution: Applying Newton's second law to masses 1 and 2 yields:

$$
\begin{aligned}
& m_{1} a_{1}=-m_{1} \omega^{2} \cdot d=-T_{A}+T_{B} \\
& m_{2} a_{2}=-m_{2} \omega^{2} \cdot(2 d)=-T_{B}
\end{aligned}
$$

where we have used that the masses have a common angular acceleration, $\alpha=\omega^{2}$, to express the left-hand side in terms of the angular rotation $\omega$. It follows that

$$
\begin{array}{ll}
+5 & \begin{array}{l}
T_{B}= \\
T_{A}= \\
+5
\end{array} \\
\left.\left.\begin{array}{l}
T_{2} \omega^{2} d \\
T_{B}-T_{A}=-m_{1} \omega^{2} d \\
\frac{T_{B}}{-}=-2 m_{2} \omega^{2} d
\end{array}\right\} \Rightarrow-m_{1}\right) \omega^{2} d \\
& \Rightarrow-T_{A}=-m_{1} \omega^{2} d-2 m_{2} \omega^{2} d \\
& \Rightarrow T A=\omega^{2} d\left(m,+2 m_{2}\right)
\end{array}
$$

A3.
Solution:


$$
\begin{aligned}
L & =T-U \\
& =\frac{1}{2} m \dot{x}^{2}-U(x) \\
& =\frac{1}{2} m \dot{x}^{2}-\frac{k}{x} e^{-t / \tau},
\end{aligned}
$$

where the potential $U(x)$ is determined from $F(x, t)$,


$$
U(x)=-\int F d x=\frac{k}{x} e^{-t / \tau}
$$

The Hamiltonian is $H=p_{x} \dot{x}-L$ with canonical momentum given by

$$
+5 \quad p_{x}=\frac{\partial L}{\partial \dot{x}}=m \dot{x}
$$

Substituting and simplifying yields

$$
\begin{aligned}
H & =p_{x} \dot{x}-L \\
& =\frac{p_{x}^{2}}{2 m}+\frac{k}{x} e^{-t / \tau}
\end{aligned} \quad \begin{aligned}
\text { in terms } \dot{x} p \text { not }
\end{aligned}
$$

The Hamiltonian is the total energy. $\frac{\partial L}{\partial t} \neq 0$, thus energy is not conserved.
$\square$
(1) May Preliminary Exam problems By Maral Ngoko classical
pb \#1: Easy - Mechanics
Alder the critical uicumstance that the astronaut just starts to slide, one has:

$$
f_{s}=m a_{n}
$$

Where $f_{S}=\mu_{S} N=\mu_{S} m g$ is the static friction

$$
\begin{aligned}
& f_{S}=\mu_{S} N=\mu_{S} m \\
& a_{n}=\frac{V^{2}}{R}=R \omega^{2} \text { is the centripetal acceluation }
\end{aligned}
$$

$+10$

$\Rightarrow \quad m R \omega^{2}=\mu_{s} m g$
$\Rightarrow R=\frac{\mu_{s} g}{\omega^{2}}$ where $g$ on that planet has to be determined.
$+10$
$v_{0 y}!_{0}$

$$
\begin{aligned}
& v_{y}=v_{0 y}+a_{y} t \Rightarrow a_{y} \operatorname{toq}=\frac{v_{y}-v_{0 y}}{t} \\
& =\frac{-30 \mathrm{~m} / \mathrm{s}) \times 2}{20 \mathrm{~s}} \\
& \Rightarrow g=\left|a_{y}\right|=3.0 \mathrm{~m} / \mathrm{s}^{2} \\
& R=\frac{0.4 \times 165}{2^{2}}=0.30 \mathrm{~m}=30 \mathrm{um}+5
\end{aligned}
$$

B1.
Solution: The position of the stone as a function of time is

$$
f<t \quad s(t)=h-\frac{1}{2} g t^{2}
$$

while the position of the rocket can be determined starting from $a_{y}(t)$ :

$$
f \begin{aligned}
a_{y}(t) & =A-B t \\
v_{y}(t) & =\int_{0}^{t} a_{y}\left(t^{\prime}\right) d t^{\prime}=A t-\frac{1}{2} B t^{2} \\
r(t) & =\int_{0}^{t} v_{y}\left(t^{\prime}\right) d t^{\prime}=\frac{1}{2} A t^{2}-\frac{1}{6} B t^{3}
\end{aligned}
$$

The maximum height the rocket attains occurs at time $T$ when $v_{y}(T)=0$.

$$
\begin{aligned}
v_{y}(T) & =0=T\left(A-\frac{1}{2} B T\right) \\
T & =0, \frac{2 A}{B}
\end{aligned}
$$

We want the latter (non-zero) time. The initial height $h$ can be solved by equating $s(t)=r(t)$ at time $t=T$.

$$
\begin{aligned}
r(T) & =r_{\max }=\frac{2 A^{3}}{3 B^{2}} \\
s(T) & =h-\frac{1}{2} g\left(\frac{2 A}{B}\right)^{2} \\
h & =\frac{2 A^{3}}{3 B^{2}}+\frac{1}{2} g\left(\frac{2 A}{B}\right)^{2} \\
& =\frac{2 A^{2}}{B^{2}}\left(g+\frac{A}{3}\right)
\end{aligned}
$$

B2.
Solution: For simple projectile motion under constant acceleration and ignoring air resistance, the equations of motions can be solved yielding


$$
\begin{aligned}
x(t) & =v_{0} \cos \theta t \\
y(t) & =v_{0} \sin \theta t-\frac{1}{2} g t^{2}
\end{aligned}
$$

where $v_{0}$ is the initial speed of the projectile, $\theta$ is the angle above the horizontal at which the projectile is fired, and $g$ is the constant acceleration due to gravity near the surface of the Earth. The distance of the projectile from the launch point is

$$
\begin{aligned}
d^{2} & =x^{2}(t)+y^{2}(t) \\
& =t^{2}\left[v_{0}^{2}-v_{0} \sin \theta g t+\left(\frac{g t}{2}\right)^{2}\right]
\end{aligned}
$$

The distance from the origin increases at least until the projectile reaches its maximum height. Thus, $\frac{d^{2}}{d t^{2}}\left(d^{2}\right)>0$ shortly after $t=0$ and the distance increases until $\frac{d^{2}}{d t^{2}}\left(d^{2}\right)$ becomes negative. So, let's find the turning point, i.e. when $\frac{d^{2}}{d t^{2}}\left(d^{2}\right)=0$.

$$
\begin{aligned}
\frac{d^{2}}{d t^{2}}\left(d^{2}\right) & =2 v_{0}^{2} t-3 v_{0} \sin \theta g t^{2}+g^{2} t^{3}=0 \\
& =t\left(2 v_{0}^{2}-3 v_{0} \sin \theta g t+g^{2} t^{2}\right) \\
t & =0, \frac{3 v_{0} \sin \theta}{2 g} \pm \frac{v_{0}}{2 g} \sqrt{9 \sin ^{2} \theta-8}
\end{aligned}
$$

The first solution is trivial so let's focus on the others. These solutions are real only when the term under the square root is non-negative, i.e.

$$
\begin{aligned}
9 \sin ^{2} \theta-8 & \geq 0 \\
\sin \theta & \geq \frac{2 \sqrt{2}}{3}
\end{aligned}
$$

Imaginary time is unphysical and thus for

there is no turning point and $\frac{d^{2}}{d t^{2}}\left(d^{2}\right)>0$.

BS.
BS

(a) for rolling spikier $2^{\text {nd }}$ Newton law

$$
\frac{m v^{2}}{R+a}=m g \cos \theta-N
$$

where $N$ is normal reaction
when the sphere poes off $N=0$
$+10$
$+10$
(6)

$$
\begin{aligned}
& m g(R+a)(1-\cos \theta)=\frac{m v^{2}}{2}+\frac{I \omega^{2}}{2} \\
& \omega=\frac{v}{a}(12 \theta \operatorname{sinping}) \quad I=\frac{2}{5} m a^{2} \\
& m g(R+a)(1-\cos \theta)=\frac{m}{2}\left(v^{2}+\frac{2}{5} v^{2}\right) \\
& d r o m(1) \frac{v}{R+a}=g \cos \theta \\
& g(1-\cos \theta)=g \frac{1}{2} \cos \theta=\frac{7}{5} \\
& \cos \theta=\frac{1}{1+\frac{7}{10}}=\frac{10}{17} \quad \theta=54,0
\end{aligned}
$$

Accept $\theta=\cos ^{-1}\left(\frac{10}{17}\right) r \cos \theta=\frac{10}{17}$.

$$
L=T-V=\frac{m}{2}\left(v^{2}+\frac{2}{5} v^{2}\right)-m g\left(R+c_{2}\right)(1-\cos \theta)
$$

Choosing $\theta$ as a generalized coordinate, $U=Q \dot{\theta}$


$$
L=\frac{7}{10} m R^{2} \dot{\theta}^{2}-\ln y(R+a)(1-\cos \theta)
$$

B4 problem \#2: hard classical mechanics

A) velocity of the cm of the collins
 cylinder

$$
\left.\begin{array}{c}
\left\{\begin{array}{l}
x=(R-r) \sin \theta \\
y=-(R-r) \cos \theta
\end{array}\right.  \tag{1}\\
+S \Rightarrow\left\{\begin{array}{l}
\dot{x}=(R-r) \dot{\theta} \cos \theta \\
\dot{y}=(R-r) \dot{\theta} \sin \theta
\end{array} \quad v^{2}=(R-r)^{2} \dot{\theta}^{2}\right.
\end{array}\right\}
$$

$$
\begin{align*}
& . V=+m g y=-m g(R)  \tag{6}\\
& +L=\frac{1}{2} m(R-r)^{2} \dot{\theta}^{2}+\frac{1}{4} m r^{2} \dot{\phi}^{2}+m g(R-r) \cos \theta
\end{align*}
$$

Also accept $\phi$ written in terms of $\dot{\theta}$ using the no-slip Condition

Condition for rothing withour supping

$$
\begin{equation*}
(R-r) \theta=r \phi \Longleftrightarrow(R-r) \dot{\theta}=r \dot{\phi} \tag{7}
\end{equation*}
$$

Lagrange's equation of motion wrt $\theta$.
$(7)$ in $(6) \Leftrightarrow L=\frac{1}{2} m(R-r)^{2} \dot{\theta}^{2}+\frac{1}{4} m r^{2} \frac{(R-r)^{2} \dot{\theta}^{2}}{r^{2}}$

$$
+m g(R-r) \cos \theta
$$

$$
\begin{equation*}
L=\frac{3}{4} m(R-r)^{2} \dot{\theta}^{2}+m g(R-r) \cos \theta \tag{8}
\end{equation*}
$$

equation of motion: $\frac{d}{d t}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=0$

- $\frac{\partial L}{\partial \theta}=-m g(R-r) \sin \theta$
- $\frac{\partial L}{\partial \dot{\theta}}=\frac{3}{2} m(R-r)^{2} \dot{\theta}$
(10) and (11) in (9) lead to :

$$
\begin{align*}
& \text { (10) and (11) in (9) lead } 2, m(R-r) \operatorname{anc} \theta=0 \\
& \frac{3}{2} m(R-r)^{2} \ddot{\theta}+m g\left(\ddot{\theta}+\frac{2}{3}\left(\frac{g}{R-r}\right) \sin \theta=0\right. \tag{12}
\end{align*}
$$

d) Small osullations $\theta \ll 1 \Rightarrow \sin \theta \approx \theta(\mathrm{rad}$.

$$
\Rightarrow \ddot{\theta}+\frac{2}{3}\left(\frac{g}{R-r}\right) \theta=0
$$

equation
characturitic of sample harmonic
with Singular frequency $\omega^{2}=\frac{2}{3} \frac{g}{R-r}$ and thus period:
$+5$

$$
\begin{gathered}
T=\frac{2 \pi}{\omega}=2 \pi \sqrt{\frac{3(R-r)}{2 g}} \\
T=\pi \sqrt{\frac{6(R-r)}{g}}
\end{gathered}
$$

Also accept approximating the Lagrangian for small $\theta$ and then finding E-L equations

A1. A 600 g copper ball has a temperature of $700^{\circ} \mathrm{C}$ when it is placed in 3.00 kg of water at a temperature of $20^{\circ} \mathrm{C}$. Calculate the temperature (in ${ }^{\circ} \mathrm{C}$ ) of the system when equilibrium has been reached? Assume the system is thermally insulated. Data: $C_{\text {water }}=4.18 \mathrm{~J} \mathrm{~g}^{-1} \mathrm{~K}^{-1}, C_{\mathrm{Cu}}=0.39 \mathrm{~J} \mathrm{~g}^{-1} \mathrm{~K}^{-1}$

Answer
Heat entering water = heat leaving copper

$$
\text { }\} 6 \text { points }
$$




A2. One mole of diatomic ideal gas ( $C_{V}=2.5 \mathrm{nR}$ ) performs a transformation from an initial state for which temperature and volume are, 290 K and 30000 ml to a final state in which temperature and volume are 310 K and 16000 ml . The transformation is represented on the (V, P) diagram by a straight line. Find the work performed and the heat absorbed by the system.

Solution:
A straight line in $(\mathrm{P}, \mathrm{V})$ diagram: $P=P 1+\frac{(V-V 1) \Delta P}{\Delta V}, \xi 6$ point $s$ $\left.W=\int P d V=\int\left[P 1+\frac{(V-V 1) \Delta P}{\Delta V}\right] d V=\left[P 1-\frac{V_{1} \Delta P}{\Delta V}\right] \Delta V+\frac{1}{2} \Delta P(V 1+V 2)=-1690 \mathrm{~J}\right\}$ 文 points $\left.\begin{array}{l}\begin{array}{l}\Delta U=C V \Delta T=416 J \\ Q=\Delta U+W=-1275 J\end{array}\end{array}\right\} 12$ points

A3. Consider a gas with the following speed probability distribution $f(v)=A$ when $0<v<v_{0}$, $=0$ otherwise. Find (a) average speed, (b) rms speed in the 1 dimensional, 2 dimensional, and 3 dimensional cases.

Solution:
1D: $\left.\langle v\rangle=\frac{\int f(v) v d v}{\int f(v) d v}=v_{0} / 2, v_{r m s}^{2}=\frac{\int f(v) v^{2} d v}{\int f(v) d v}=v_{0}^{2} / 3\right\} \&$ points
aD: $\left.\langle v\rangle=\frac{\int f(v) v d^{2} v}{\int f(v) d^{2} v}=\frac{2}{3} v_{0}, v_{r m s}^{2}=\frac{\int f(v) v^{2} d^{2} v}{\int f\left(v d^{2} v\right.}=v_{0}^{2} / 2\right\} 8$ points
3D: $\left.\langle v\rangle=\frac{\int f(v) v d^{3} v}{\int f(v) d^{3} v}=3 / 4 v_{0}, v_{r m s}^{2}=\frac{\int f(v) v^{2} d^{3} v}{\int f(v) d^{3} v}=v_{0}^{2} 3 / 5\right\} 8$ points +1point if all coefficients are correct

A4. An ideal gas ( $\gamma=1.4$ ) expand in an adiabatic process to 10 times of its original volume. If the initial temperature is $0^{\circ} \mathrm{C}$, and the initial pressure 1 atm find the final temperature.

## Solution:

Adiabatic equation 12 points

$$
\underbrace{\overbrace{\text { The }}^{\mathrm{T}=\mathrm{T}_{0} 10^{1-\gamma}=108 \text { K. PeP P } / 10^{\prime \prime}=0.0398 \mathrm{~atm}}}_{\text {correct solution }}
$$

B1. An ideal gas is expanded adiabatically from $\left(p_{1}, V_{1}\right)$ to $\left(p_{2}, V_{2}\right)$. It is then compressed isobarically to ( $p_{2}, V_{1}$ ). Finally, the pressure is increased to $p_{1}$ at constant volume $V_{1}$. Show that the efficiency of the cycle is

$$
\eta=1-\gamma \frac{V_{2} / V_{1}-1}{p_{1} / p_{2}-1}
$$

Answer


The work the system does in the cycle is

$$
\left.W=\oint p d V=\int_{A B} p d V+p_{2}\left(V_{1}-V_{2}\right) .\right\} 10 \text { points }
$$

Because $A B$ is adiabatic and an ideal gas has the equations $p V=n k T$ and $C_{p}=C_{v}+R$, we get

$$
\left.\begin{array}{rl}
\int_{A B} p d V & =-\int_{A B} C_{v} d T=-C_{v}\left(T_{2}-T_{1}\right) \\
& =\frac{1}{1-\sim}\left(p_{2} V_{2}-p_{1} V_{1}\right) .
\end{array}\right\} 10 p 0 i n t
$$

During the $C A$ part of the cycle the gas absorbs heat

$$
\begin{aligned}
Q & =\int_{C A} T d S=\int_{C A} C_{v} d T=C_{v}\left(T_{1}-T_{2}\right) \\
& =\frac{1}{1-\gamma} V_{1}\left(p_{2}-p_{1}\right)
\end{aligned}
$$

Hence, the efficiency of the engine is

$$
\eta=\frac{W}{Q}=1-\gamma \frac{\frac{V_{2}}{V_{1}}-1}{\frac{p_{1}}{p_{2}}-1}
$$

B2. The entropy of an ideal gas is $S=n / 2[a+5 R \ln (U / n)+2 R \ln (V / n)]$, where $n$ is the mole number, $R$ is the universal gas constant, $U$ is internal energy, $V$ is volume, and $a$ is a constant.
(a) Calculate the constant pressure heat capacity $\left(C_{P}\right)$ and the constant volume heat capacity $\left(C_{V}\right)$.
(b) Rewrite entropy in ( $\mathrm{T}, \mathrm{V}$ ), ( $\mathrm{T}, \mathrm{P}$ ), and ( $\mathrm{P}, \mathrm{V}$ ) representation.

$$
\begin{aligned}
& \text { Solution: } \\
& \qquad \begin{array}{l}
\left.C_{V}=T\left(\frac{\partial S}{\partial T}\right)_{V}=T \frac{n}{2} \frac{5 R}{U}\left(\frac{\partial U}{\partial T}\right)_{V}=\frac{5}{2} \frac{n R T}{U} C_{V}\right\} G p \text { int } \\
\text { Therefore, } U=\frac{5}{2} n R T . \\
C_{P}=C_{V}=\frac{5}{2} n R
\end{array}
\end{aligned}
$$

$$
S(T, V)=n c_{V} \ln \frac{T}{T_{o}}+n R \ln \frac{V}{V_{o}}+n s_{o} . \quad \text { y point } \zeta
$$

$$
S(T, P)=n c_{P} \ln \frac{T}{T_{o}}-n R \ln \frac{P}{P_{o}}+n s_{o} . \quad 4 \text { points }
$$

$$
S(P, V)=n c_{V} \ln \frac{P}{P_{o}}+n c_{P} \ln \frac{V}{V_{o}}+n s_{o} . \text { Y point } 5
$$

Also accept without $P_{0}, V_{0}$, and $T_{0}$.
B3. Consider the adiabatic free expansion of an ideal gas (from volume V to 2 V ).
(a) What's the work and heat in the process?
(b) How does the temperature change?
(c) Show that this process is irreversible. (Hint: calculate the entropy difference)
(d) answer (b) if the gas is nonideal.

## Solution:


(a) In this process, there is no work and heat. The internal energy does not change, or the temperature does not change.
(b) Suppose the initial state of the gas can be described as $\mathrm{P}, \mathrm{V}, \mathrm{T}$, the final state of gas can be described as $\mathrm{P} / 2,2 \mathrm{~V}, \mathrm{~T}$.
(c) So, one can use an isothermal process (reversible) to bring the system from the initial to the final state, and calculate the entropy change using that process.

Since dU =0,

$$
\begin{aligned}
\Delta S=\frac{1}{T} \int(d U+P d V) & =\frac{1}{T} \int P d V=\frac{1}{T} \int\left(\frac{n R T}{V}\right) d V=n R \ln (2 V / V) \\
=n R \ln 2 & >0 .
\end{aligned}
$$

Because this process is adiabatic and $\Delta S>0$, it is irreversible.
$\int$ (d) If the gas is not ideal, the potential energy of interaction between molecules increases during the expansion, therefore due to conservation of energy the kinetic energy decreases, therefore the temperature slightly decreases

BU.


$$
p_{0} V_{0}=\frac{f_{2}}{2} V_{1} \rightarrow V_{1}=2 V_{0}
$$



$$
\begin{aligned}
& f_{0}\left(2 V_{0}\right)^{\gamma}=1.32 p_{0} V_{0}^{\gamma} \\
& \frac{1}{2} 2^{\gamma}=1.32 \\
& \gamma^{\prime}=\frac{\ln 2.64}{\ln 2}=1.4
\end{aligned}
$$

for a gas with $s$ degrees of freedom

$$
\gamma=\frac{\frac{S}{2}+1}{\frac{S}{2}}=1+\frac{2}{S}=1.4 \rightarrow S=5
$$

$\longrightarrow g a s$ is diatomic
(C) Cook at the temperature


$$
T V^{J^{\prime}-1}=\text { cons }
$$

$$
\frac{T_{2}}{T_{0}}=\left(\frac{R V_{0}}{V_{0}}\right)^{x-1} \rightarrow T_{2}=T_{0} 2^{x-1}=2^{0.4} T_{0}=1.32 T_{0}
$$

the translational kinetic energy $=\frac{3}{2} k_{B} T$ increases

