**A1.** A particle of charge $q$ is moved from infinity into the center of a hollow conducting spherical shell of inner radius $a$ and thickness $t$, through a tiny hole in the shell. How much work is required?

**Solution:**

The work done by the external force is equal to the increase of the electrostatic energy of the whole system. The electric field intensity at point **r** from the charge $q$ is $= \frac{q}{4\pi\varepsilon_0 r^2}$ . When $q$ is at infinity, the electrostatic energy of the whole system is $= \int \frac{\varepsilon_0}{2} E^2 d^3 r$ , where the integration is over all space. This because the distance between the spherical shell and charge $q$ is infinite so that the field at the conducting sphere can be taken to be zero. After $q$ has been moved to the center of the conducting spherical shell, as the shell has no effect on the field inside, the electric intensity inside the shell is still $E = \frac{q}{4\pi\varepsilon_0 r^2}$, where $r$ is the distance from charge $q$. Outside the shell, Gauss' law says that the electric intensity is still $E = \frac{q}{4\pi\varepsilon_0 r^2}$. Hence the electrostatic energy of the system remains the same as $U$ but minus the contribution of the shell itself, inside whose thickness the field is zero. Thus, a decrease of the electrostatic energy is given by

$$\Delta U = \int_a^{a+t} \frac{\varepsilon_0}{2} \left( \frac{q}{4\pi\varepsilon_0 r^2} \right)^2 4\pi r^2 d^3 r = \frac{q^2}{8\pi\varepsilon_0} \left( \frac{1}{a} - \frac{1}{a+t} \right) \ ,$$

and equal to the negative work done by the external force.

**A2**. A slab of homogeneous dielectric material of dielectric permittivity $\varepsilon$ and thickness $d$ is infinite in the $z$ plane. It is placed in an external field $\mathbf{E}_0 = E_0\mathbf{z}$, where $E_0$ is a constant. There are no free charges in the slab. Using the electrostatic boundary conditions, find the electric field and induced polarization charge density $\sigma_P$ on top and bottom surfaces of the slab. Find the electric field $\mathbf{E}_P$ which is produced by the polarization charges and show that $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_P$.

**Solution:**

Since there are no free surface charges, according to the electrostatic boundary conditions, the normal component of the electric displacement **D** is continuous across the surface of the slab. Outside the slab $\mathbf{D} = \varepsilon_0\mathbf{E}_0$, whereas inside the slab $\mathbf{D} = \varepsilon\mathbf{E}$, where $\mathbf{E}$ is the field in the dielectric.

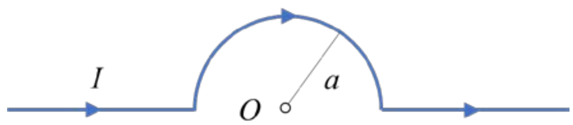
From the continuity of **D**, we obtain $\mathbf{E} = \frac{\varepsilon_0}{\varepsilon}\mathbf{E}_0$ .

In the absence of free charges, the normal component of the electric field has a step of $\frac{\sigma_P}{\varepsilon_0}$ when crossing the surface. Since the normal component of the electric field is $E_0$ outside the dielectric and $E = \frac{\varepsilon_0}{\varepsilon}E_0$ inside the dielectric, we obtain $\sigma_P = \sigma_0 = \varepsilon_0(\frac{\varepsilon_0}{\varepsilon} - 1)E_0$ on the bottom surface and $\sigma_P = -\sigma_0 = -\varepsilon_0(\frac{\varepsilon_0}{\varepsilon} - 1)E_0$ on the top surface.

The surface polarization charges create a capacitor-like structure, so that $\mathbf{E}_P = \frac{\sigma_0}{\varepsilon_0}\mathbf{z} = (\frac{\varepsilon_0}{\varepsilon} - 1)\mathbf{E}_0$.

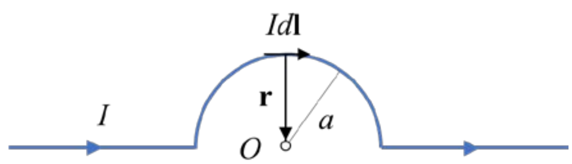
Since $\mathbf{E} = \frac{\varepsilon_0}{\varepsilon}\mathbf{E}_0$, it is easy to see that $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_P$.

**A3.** An infinitely long wire carries a current $I$ = 1 A. It is bent so to have a semi-circular detour around the origin with radius $a$ = 1 cm, as shown in the figure below. Calculate the magnetic field at the origin.




**Solution:**

The straight parts of the wire do not contribute to the magnetic field at $O$ since for them $Id\mathbf{l}\times\mathbf{r}=0$ (see figure below).




We need only to consider the contribution of the semi-circular part. The magnetic field at $O$ produced by a current element Id**l** is

$$d\mathbf{B}=\frac{\mu_0}{4\pi}\frac{Id\mathbf{l}\times\mathbf{r}}{r^3}$$

As Id**l** and **r** are mutually perpendicular for the semi-circular wire, $d\mathbf{B}$ is always pointing into the page. The total magnetic field of the semi-circular wire is then

$$B=\int dB=\frac{\mu_0 I}{4\pi a}\int_0^\pi d\theta=\frac{\mu_0 I}{4a}.$$

With $I$ = 1A, $a$ =$10^{-2}$ m, the magnetic induction at $O$ is $B$ = 3.14 x $10^{-5}$ T, pointing perpendicularly into the page.

A4. Employ the Faraday law in the form

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}.$$

The magnetc flux inside the loop

$$\Phi = B\pi r^2 = \mu_0 n I \pi r^2$$

for $r < a$, and $\mu_0 n I \pi a^2$ for $r > a$, therefore

$$\frac{d\Phi}{dt} = \pi \mu_0 n b r^2$$

inside and $\pi \mu_0 n b a^2$ outside. Since the magnetic field is straight, the electric field is circumferential, therefore the l.h.s. is $2\pi r E$ and

$$E = -\frac{1}{2}\mu_0 r n b$$

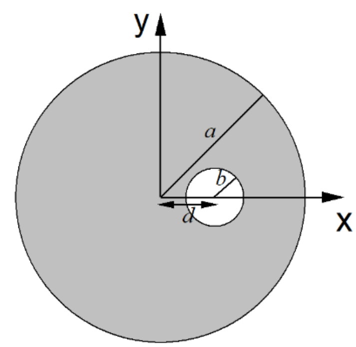
inside and

$$E = -\frac{a^2}{2r}\mu_0 n b$$

outside.

Its direction is opposite to the direction of the growing current (the Lenz rule).

**B1.** Consider an infinite cylindrical wire oriented along the $z$ direction with radius $a$. This wire has an infinite cylindrical cavity parallel to the wire with radius $b$, but displaced from the axis by a distance $d$ along the $x$ direction (see the cross-section of the wire in the figure below). This wire carries a total current $I$ uniformly distributed throughout its cross-section flowing along the $+z$ direction. Using Ampere's law and the superposition principle find the magnetic field inside the cavity.




**Solution:**

First, we note that the current density, $J$, flowing in the wire is given by the total current divided by the cross-sectional area of the wire:

$$\mathbf{J} = \frac{I}{\pi\left(a^2 - b^2\right)}\hat{\mathbf{z}} \qquad (1)$$

We will use the superposition principle in the following way. First, we find the magnetic field, $\mathbf{B}_1$, inside a solid wire (i.e. without a cavity) of radius $a$ carrying uniform current density $\mathbf{J}$. Next, we find the field, $\mathbf{B}_2$, generated by a solid wire of radius $b$ displaced along $x$ also carrying uniform current density, but equal and opposite to the larger wire, i.e. a current density $-\mathbf{J}$. By the superposition principle the sum of these two fields, $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2$, is the field of the original problem.

$\mathbf{B}_1$ is easy to find using Ampere's law. Because of the axial symmetry and using the right-hand rule we know the field will be independent of $\varphi$ and directed along $\hat{\varphi}$. We therefore take an Amperian loop of radius $s < a$ and find that

$$2\pi s B_1 = \mu_0 \pi s^2 J \Rightarrow \mathbf{B}_1 = \frac{\mu_0 J s}{2}\hat{\varphi}. \qquad (2)$$

Rewriting this in Cartesian coordinates we find

$$\begin{aligned} \mathbf{B}_1 &= \frac{\mu_0 J}{2}\sqrt{x^2+y^2}\left[-\sin\varphi\hat{\mathbf{x}} + \cos\varphi\hat{\mathbf{y}}\right] \\ &= -\frac{\mu_0 J}{2}\left[y\hat{\mathbf{x}} - x\hat{\mathbf{y}}\right] \end{aligned} \qquad (3)$$

This is the field inside the larger wire. Now we need the field generated inside the smaller wire by current $-\mathbf{J}$. This is given by a formula similar to (3) but with current reversed (i.e. $J \to -J$) and axis shifted along the $x$ direction (i.e. $x \to x - d$):

$$\mathbf{B}_2 = \frac{\mu_0 J}{2}\left[y\hat{\mathbf{x}} - \left(x-d\right)\hat{\mathbf{y}}\right]. \qquad (4)$$

Adding (3) and (4) we therefore find the total field inside the cavity of the original problem

$$\mathbf{B} = -\frac{\mu_0 J}{2}\left[y\hat{\mathbf{x}} - x\hat{\mathbf{y}}\right] + \frac{\mu_0 J}{2}\left[y\hat{\mathbf{x}} - \left(x-d\right)\hat{\mathbf{y}}\right] = \frac{\mu_0 J d}{2}\hat{\mathbf{y}} = \frac{\mu_0 I d}{2\pi\left(a^2-b^2\right)}\hat{\mathbf{y}}. \qquad (5)$$

It is seen that the field is uniform and oriented along the $y$ direction.

**B2.** A grounded spherical metal shell of radius R is filled with a space charge of uniform charge density $\rho$. Find the electric field, the electric potential, and the electrostatic energy of the system.

**Solution:**

Consider a concentric spherical surface of radius $r$ ($r < R$). Using Gauss' law we obtain

$$\mathbf{E}(\mathbf{r}) = \frac{\rho r}{3\varepsilon_0}\hat{\mathbf{r}}\,.$$

As the shell is grounded, $\Phi(R) = 0$, $E(r) = 0$ for $r > R$.

Thus, we obtain for the potential: $\Phi(r) = \int_r^R E(r)dr = \frac{\rho}{6\varepsilon_0}(R^2 - r^2)$

The electrostatic energy is $W = \frac{1}{2}\int \rho\Phi d^3r = \frac{1}{2}\int_0^R \frac{\rho^2}{6\varepsilon_0}(R^2 - r^2)4\pi r^2 dr = \frac{2\rho^2 R^5}{45\varepsilon_0}$ .

B2 solution (IF version). In Evgeny's solution I don't understand why E=0 outside the sphere. So I used the Poisson equation approach.

Inside the sphere we have

$$\frac{1}{r}\frac{d^2}{dr^2}(r\phi) = -\frac{\rho}{\epsilon_0}$$

with the boundary conditions $\phi(0)$ is finite, and $\phi(R) = 0$. Integrating this equation twice, we obtain

$$\phi(r) = \frac{\rho}{6\epsilon_0}(R^2 - r^2).$$

Outside the sphere

$$\frac{1}{r}\frac{d^2}{dr^2}(r\phi) = 0$$

with the boundary conditions $\phi(R) = 0$, $\phi(\infty) = 0$. This gives

$$\phi(r) \equiv 0.$$

From here we obtain

$$E(r) = -\frac{d\phi}{dr} = \frac{\rho}{3\epsilon_0}r$$

for $r < R$, and $E(r) = 0$ for $r > R$.

**B3:** A spherical conductor of uniform conductivity $\sigma$ has a uniform volume charge density $\rho_0$ and time t = 0. Describe time evolution of the electric field and electric current in the conductor. Discuss what happens at $t \to \infty$.

**Solution:**

From the Gauss' law $\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$, the continuity equation $\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$, and the definition of conductivity $\mathbf{J} = \sigma \mathbf{E}$, we obtain:

$$\frac{\partial \rho}{\partial t} = -\frac{\sigma}{\varepsilon_0}\rho ,$$

resulting in

$$\rho(t) = \rho_0 e^{-\frac{\sigma}{\varepsilon_0}t} \text{ and } \nabla \cdot \mathbf{E} = \frac{\rho_0}{\varepsilon_0} e^{-\frac{\sigma}{\varepsilon_0}t} .$$

The spherical symmetry requires that $\mathbf{E}(\mathbf{r}) = E(r)\hat{\mathbf{r}}$. Hence we obtain:

$$\nabla \cdot \mathbf{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E(r)\right) = \frac{\rho_0}{\varepsilon_0} e^{-\frac{\sigma}{\varepsilon_0}t} ,$$

giving

$$\mathbf{E}(\mathbf{r},t) = \frac{\rho_0 \mathbf{r}}{3\varepsilon_0} e^{-\frac{\sigma}{\varepsilon_0}t} + \mathbf{E}(0,t) = \frac{\rho_0 \mathbf{r}}{3\varepsilon_0} e^{-\frac{\sigma}{\varepsilon_0}t}$$

and

$$\mathbf{J} = \frac{\sigma \rho_0 r}{3\varepsilon_0} e^{-\frac{\sigma}{\varepsilon_0}t} \hat{\mathbf{r}}$$

Note that $\mathbf{E}(0,t) = 0$ by symmetry. It is evident that at $t \to \infty$, $\mathbf{E} = 0$, $J = 0$, and $\rho = 0$ inside the conductor. Thus, the charge is uniformly distributed on the spherical surface and there are no fields, charges, and current inside the conductor after a sufficiently large time.

B4. El. field:

(a) $\vec{E} = E_0 \hat{n} \cos(\vec{k}\cdot\vec{r} - \omega t)$

$\hat{k} = \frac{1}{\sqrt{3}}(1,1,1) \qquad \vec{k} = \frac{\omega}{c\sqrt{3}}(1,1,1)$

$\hat{n} = \hat{x}\cos\alpha + \hat{z}\sin\alpha \qquad$ since $|\hat{n}| = 1$

from $\hat{n}\cdot\hat{k} = 0$ we obtain $\cos\alpha + \sin\alpha = 0$, $\alpha = -\frac{\pi}{4}$

$\hat{n} = \frac{1}{\sqrt{2}}(\hat{x} - \hat{z})$

(b) $\vec{B} = \frac{1}{c}\hat{k}\times\vec{E} = \frac{1}{c}(\hat{k}\times\hat{n})E_0\cos(\vec{k}\cdot\vec{r} - \omega t)$

$\hat{k}\times\hat{n} = \frac{1}{\sqrt{6}}\begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{6}}(-\hat{x} + 2\hat{y} - \hat{z}) \equiv \hat{n}_1$

(c) $\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \frac{E_0^2}{\mu_0 c}(\hat{n}\times\hat{n}_1)\cos^2(\vec{k}\cdot\vec{r} - \omega t)$

$\hat{n}\times\hat{n}_1 = \frac{1}{\sqrt{12}}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 0 & -1 \\ -1 & 2 & -1 \end{vmatrix} = \frac{1}{\sqrt{12}}(2\hat{x} + 2\hat{y} + 2\hat{z}) = \frac{1}{\sqrt{3}}(\hat{x} + \hat{y} + \hat{z}) = \hat{k}$

$\vec{S} = \frac{E_0^2}{\mu_0 c}\hat{k}\cos^2(\vec{k}\cdot\vec{r} - \omega t) = \varepsilon_0 c E_0^2 \hat{k}\cos^2(\vec{k}\cdot\vec{r} - \omega t)$

where we used $\varepsilon_0\mu_0 c^2 = 1$

(d) $u = \frac{|\vec{S}|}{c} = \cancel{\varepsilon_0 c} \; \varepsilon_0 E_0^2 \cos^2(\vec{k}\cdot\vec{r} - \omega t)$

**A1.**

hc/λ=φ+K

(a) K=V=1.10 eV
**(b)** φ= hc/λ-K= 1240 eV.nm/400nm-1.10 eV=2.00 eV
**(c)** $\lambda_{cut}$= hc/φ=620 nm
**(d)**

**Problem A2**

Operator *R* is defined by $R\psi(x) = \mathrm{Re}[\psi(x)]$. Is *R* Hermitian?

ANSWER

The operator is Hermitian when $\langle \varphi | R\psi \rangle = \langle R\varphi | \psi \rangle$. We calculate LHS and RHS:

$$\langle \varphi | R\psi \rangle = \int \varphi^*(x) R\psi(x) dx = \int \varphi^*(x) \mathrm{Re}\left[\psi(x)\right] dx =$$
$$\int \varphi^*(x) \tfrac{1}{2}\left[\psi(x) + \psi^*(x)\right] dx = \tfrac{1}{2}\int \varphi^*(x)\psi(x)dx + \tfrac{1}{2}\int \varphi^*(x)\psi^*(x)dx$$

$$\langle R\varphi | \psi \rangle = \int (R\varphi(x))^* \psi(x) dx = \int \left[\mathrm{Re}\,\varphi(x)\right]^* \psi(x) dx =$$
$$\tfrac{1}{2}\int \left[\varphi(x) + \varphi^*(x)\right]^* \psi(x) dx = \tfrac{1}{2}\int \varphi^*(x)\psi(x)dx + \tfrac{1}{2}\int \varphi(x)\psi(x)dx$$

So $\langle \varphi | R\psi \rangle \neq \langle R\varphi | R \rangle \quad \Rightarrow \quad R$ is not Hermitian

**Problem A3**

Part a.

write $\alpha\,|\uparrow\rangle + \beta\,|\downarrow\rangle$ as $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$

$$F\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=\begin{pmatrix}\beta\\ \alpha\end{pmatrix}$$

Writing $|\Sigma_i\rangle=\begin{pmatrix}\beta_i\\ \alpha_i\end{pmatrix}$, we have:

$$\langle\Sigma_1|F\Sigma_2\rangle=\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}^*\begin{pmatrix}\beta_2\\ \alpha_2\end{pmatrix}=\alpha_1{}^*\beta_2+\beta_1{}^*\alpha_2$$

$$\langle F\Sigma_1|\Sigma_2\rangle=\begin{pmatrix}\beta_1\\ \alpha_1\end{pmatrix}^*\begin{pmatrix}\alpha_2\\ \beta_2\end{pmatrix}=\beta_1{}^*\alpha_2+\alpha_1{}^*\beta_2$$

$\langle\Sigma_1|F\Sigma_2\rangle=\langle F\Sigma_1|\Sigma_2\rangle \quad\Rightarrow\quad F$ is Hermitian

Part b.

$F\begin{pmatrix}1\\ 0\end{pmatrix}=\begin{pmatrix}0\\ 1\end{pmatrix}$ and $F\begin{pmatrix}0\\ 1\end{pmatrix}=\begin{pmatrix}1\\ 0\end{pmatrix}$ so $F=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

Eigenvalue equation:

$$\begin{vmatrix}-\lambda & 1\\ 1 & -\lambda\end{vmatrix}=0 \quad\Rightarrow\quad \lambda^2-1=0 \quad\Rightarrow\quad \lambda=\pm1$$

Eigenkets:

For $\lambda=+1$

$$\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=0 \quad\Rightarrow\quad \alpha=\beta$$

Eigenket is $\tfrac{1}{2}\sqrt{2}\begin{pmatrix}1\\ 1\end{pmatrix}=\tfrac{1}{2}\sqrt{2}\left(|\uparrow\rangle+|\downarrow\rangle\right)$

For $\lambda=-1$

$$\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=0 \quad\Rightarrow\quad \alpha=-\beta$$

Eigenket is $\tfrac{1}{2}\sqrt{2}\begin{pmatrix}1\\ -1\end{pmatrix}=\tfrac{1}{2}\sqrt{2}\left(|\uparrow\rangle-|\downarrow\rangle\right)$

**Problem A4**

$E_n=-kme^4/(2n^2\ hbar^2)$

Where m is the reduced mass. For hydrogen $m=m_e$, and $kme^4/(2\ hbar^2)$ =Ry (Rydberg constant)=13.6 eV

(a) For Ps $m=m_e/2$, therefore $E_n=-Ry/(2n^2)$

For n=2 $E_2=-Ry/8=-1.7$ eV

(b) $hc/\lambda=Ry/2(1/4-1/9)=5Ry/72=0.944$ eV
$\lambda=1240$ eV.nm/0.944 eV=1316 nm

**B2**. The wavefunction of a particle moving in one dimension is given by

$$\psi(x)=\begin{cases}0 & x<-b/2\\ C & -b/2<x<+b/2\\ 0 & x>+b/2\end{cases}$$

where C is a real-valued, positive constant.

a. Normalize the wavefunction.
b. Find $\varphi(k)$, the wavefunction in *k*-space ( $k = p/\hbar$ ).
c. Estimate the widths $\Delta x$ and $\Delta p$ and show they agree with Heisenberg's uncertainty principle.

<u>ANSWERS</u>

<u>Part a.</u>

$$\int_{-\infty}^{+\infty}|\psi(x)|^2\,dx=1 \quad\Rightarrow\quad \int_{-b/2}^{+b/2}C^2dx=C^2b=1 \quad\Rightarrow\quad C=\frac{1}{\sqrt{b}}$$

<u>Part b.</u>

$$\varphi(k)=\langle k\,|\,\psi\rangle=\int_{-\infty}^{+\infty}\langle k\,|\,x\rangle\langle x\,|\,\psi\rangle dx=\int_{-\infty}^{+\infty}\langle x\,|\,k\rangle^*\langle x\,|\,\psi\rangle dx=\int_{-\infty}^{+\infty}\frac{e^{-ikx}}{\sqrt{2\pi}}\psi(x)dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-ikx}\psi(x)dx$$

$$\varphi(k)=\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{b}}\int_{-b/2}^{+b/2}e^{-ikx}dx=\frac{1}{\sqrt{2\pi b}}\left[\frac{e^{-ikx}}{-ik}\right]_{-b/2}^{b/2}=\frac{i}{\sqrt{2\pi b}\,k}\left[e^{-ikb/2}-e^{ikb/2}\right]=$$

$$=\frac{-i}{\sqrt{2\pi b}\,k}\left[e^{+ikb/2}-e^{-ikb/2}\right]=\frac{2}{\sqrt{2\pi b}\,k}\sin(kb/2)=\frac{1}{\sqrt{2\pi}}\sqrt{b}\,\frac{\sin(kb/2)}{kb/2}$$

<u>Part c.</u>

For the width of $\psi(x)$ we take $\Delta x = b$.

For the width of $\varphi(k)$ we take the width between its zeroes nearest to $k=0$ :

$$\sin(kb/2)=0 \quad\Rightarrow\quad kb/2=\pm\pi \quad\Rightarrow\quad k=\pm 2\pi/b \quad\Rightarrow \Delta p=\hbar\Delta k=\hbar 4\pi/b$$

Hence, $\Delta x\Delta p = b\times\hbar 4\pi/b = 4\pi\hbar > \frac{1}{2}\hbar$

**B1**. Combine Compton formula

$$\lambda' - \lambda = \lambda_C(1 - \cos\theta)$$

with the conservation of energy

$$\frac{hc}{\lambda} = E + \frac{hc}{\lambda'}$$

where $E$ is the energy transfer to electron. Obviously, the maximum energy transfer occurs for backward scattering, $\theta = 180°$, therefore

$$\frac{hc}{\lambda} = E + \frac{hc}{\lambda + 2\lambda_C}$$

which reduces to the quadratic equation

$$\lambda^2 + 2\lambda\lambda_C - 2\mu\lambda_C = 0$$

where $\mu = hc/E = 24.8$ pm. Solve quadratic equation

$$\lambda = -\lambda_C + \sqrt{\lambda_C^2 + 2\mu\lambda_C} = -2.43 + \sqrt{2.43^2 + 2 \cdot 24.8 \cdot 2.43} = 8.814 \text{ pm}.$$

$$E_{photon} = \frac{hc}{\lambda} = 1240 \text{ keV} \cdot \text{pm}/8.814 \text{ pm} = 140.7 \text{ keV}.$$

(b)

$$\lambda' = \lambda + \lambda_C(1 - \cos 60°) = 8.814 + 2.43/2 = 10.03\text{`pm}$$

$$E = hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) = 17 \text{ keV}.$$

B3. (a)

$$Y_1^{-1}(\theta,\phi) = \Theta(\theta)\Phi(\phi),$$

$$\Phi = \frac{1}{\sqrt{2\pi}} e^{-i\phi}, \quad \Theta(\theta) = \frac{\sqrt{3}}{2}\sin\theta.$$

Since $\Phi$ is normalized, only $\theta$ integration is necessary

$$p = \int_0^{\pi/3} \Theta^2(\theta)\sin\theta d\theta = \frac{3}{4}\int_0^{\pi/3} \sin^2\theta \sin\theta d\theta.$$

with the integration variable $u = \cos\theta$

$$p = \frac{3}{4}\int_{1/2}^{1}(1-u^2)du = \frac{3}{4}\left(u - u^3/3\right)_{1/2}^{1} = \frac{3}{4}\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{8\cdot 3}\right) = \frac{5}{32}.$$

(b)

$$p = \int_{a_0}^{\infty} R_{2p}^2 r^2 dr = \frac{1}{24a_0^5}\int_{a_0}^{\infty} r^4 e^{-r/a_0} dr.$$

using the substitution $x = r/a_0$ we obtain

$$p = \frac{1}{24}\int_1^{\infty} x^4 e^{-x} dx = \frac{1}{24}e^{-1}(1+4+12+24+24) = 0.996$$

where we have used

$$\int x^4 e^{-x} dx = -e^{-x}(x^4 + 4x^3 + 12x^2 + 24x + 24).$$

(c)

$$\langle r\rangle = \int_0^{\infty} R_{2p}^2 r^3 dr = \frac{a_0^6}{24a_0^5}\int_0^{\infty} x^5 e^{-x} dx = 5a_0.$$

(d) Determine the position of the the maximum probability density

$$\frac{d}{dr}(r^4 e^{-r/a_0}) = 0$$

$$e^{-r/a_0}(4r^3 - r^4/a_0) = 0$$

$r = 4a_0$.

B4. (a) Consider the well defined as $V(x') = 0$ for $0 < x' < L$. The solution is

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{\pi n x'}{L}, n = 1, 2, ...$$

The solution for the well defined as $V(x) = 0$ for $-L/2 < x < L/2$ can be obtained by performing the transformation $x' = x + L/2$. Then

$$\psi_1(x) = \sqrt{\frac{2}{L}} \sin\left[\frac{\pi}{L}(x + L/2)\right] = \sqrt{\frac{2}{L}} \cos \frac{\pi x}{L},$$

$$\psi_2(x) = \sqrt{\frac{2}{L}} \sin\left[\frac{2\pi}{L}(x + L/2)\right] = -\sqrt{\frac{2}{L}} \sin \frac{2\pi x}{L}.$$

The sign in front is inessential.

(b)

$$\psi(x) = C[2\psi_1(x) + \psi_2(x)]$$

$$C^2(4+1) = 1, \quad C = 1/\sqrt{5}.$$

(c)

$$\langle H \rangle = \frac{1}{5} \int_{-L/2}^{L/2} (2\psi_1^* + \psi_2^*) H (2\psi_1 + \psi_2) dx.$$

$$\langle x \rangle = \frac{1}{5} \int_{-L/2}^{L/2} x |2\psi_1 + \psi_2|^2 dx.$$

Since $\psi_1$ and $\psi_2$ are eigenstates of $H$, the crossed terms in the first integral disappear, and we have

$$\langle H \rangle = \frac{1}{5}(4E_1 + E_2), \quad E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2 \pi^2 n^2}{2mL^2}$$

$$\langle H \rangle = \frac{1}{5}\frac{\hbar^2\pi^2}{2mL^2}(4+4) = \frac{4}{5}\frac{\hbar^2\pi^2}{mL^2}$$

In the second integral only the crossed terms survive since $x|\psi_1|^2$ and $x|\psi_2|^2$ are odd functions.

$$\langle x \rangle = \frac{2}{5} 2\mathrm{Re} \int_{-L/2}^{L/2} x\psi_2^*(x)\psi_1(x)dx \tag{1}$$

At $t = 0$

$$\langle x \rangle = \frac{8}{5L}\int_{-L/2}^{L/2} x \cos\frac{\pi x}{L} \sin\frac{2\pi x}{L} dx.$$

$$\int_{-L/2}^{L/2} x \cos\frac{\pi x}{L} \sin\frac{2\pi x}{L} dx = \left(\frac{L}{\pi}\right)^2 \int_{-\pi/2}^{\pi/2} y\cos(y)\sin(2y)dy = \frac{8}{9}\left(\frac{L}{\pi}\right)^2$$

Finally

$$\langle x \rangle = \frac{64L}{45\pi^2} \tag{2}$$

(d) For $t > 0$ $\psi_1$ and $\psi_2$ are multplied by $e^{-iE_1t/\hbar}$ and $e^{-iE_2t/\hbar}$, therefore from (1) and (2) we obtain

$$\langle x \rangle(t) = \frac{64L}{45\pi^2}\cos\frac{(E_2 - E_1)t}{\hbar} = \frac{64L}{45\pi^2}\cos\frac{3\pi^2\hbar t}{2mL^2}$$

Since the Hamiltonian is time-independent, $\langle H \rangle$ does not depend on time, and

$$\langle H \rangle(t) = \frac{4}{5}\frac{\hbar^2\pi^2}{mL^2}$$

for $t > 0$.