TH A1

An insurance company needs to know the probability $P(n)$ that the state of Calisota is hit by n tornadoes in a year. They find there is a probability distribution:

$$P(n) \propto \frac{\mu^n}{n!},$$

where $\mu = 2.5$.

a. What is the probability that there will be no tornadoes in Calisota next year?

b. What is the probability that Calisota will be hit by 4 or more tornadoes next year?

c. What is the average number of tornadoes per year in Calisota?

ANSWERS

a. It is convenient (though not necessary) to normalize $P(n)$ first. We require $\Sigma_n P(n) = 1$, or

$$C\,\Sigma_n \frac{\mu^n}{n!} = Ce^{\mu} = 1 \quad \Rightarrow \quad C = e^{-\mu}, \qquad \text{so } P(n) = e^{-\mu}\frac{\mu^n}{n!}.$$

Probability for no tornadoes is then $P(0) = e^{-\mu}\frac{\mu^0}{0!} = e^{-\mu} = 0.082 = 8.2\%$

b. We have

$$P(n \geq 4) = 1 - P(0) - P(1) - P(2) - P(3)$$
$$P(0) = e^{-\mu}\frac{\mu^0}{0!} = e^{-\mu}$$
$$P(0) = e^{-\mu}\frac{\mu^1}{1!} = \mu e^{-\mu}$$
$$P(0) = e^{-\mu}\frac{\mu^2}{2!} = \tfrac{1}{2}\mu^2 e^{-\mu}$$
$$P(0) = e^{-\mu}\frac{\mu^3}{3!} = \tfrac{1}{6}\mu^3 e^{-\mu}$$
$$P(0) + P(1) + P(2) + P(3) = e^{-\mu}\left(1 + \mu + \tfrac{1}{2}\mu^2 + \tfrac{1}{6}\mu^3\right) = 0.757576$$
$$P(n \geq 4) = 1 - 0.757576 = 0.242424 = 24\%$$

c. We find $\langle n \rangle = \sum_{n=0}^{\infty} nP(n) = e^{-\mu}\sum_{n=0}^{\infty} n\frac{\mu^n}{n!} = e^{-\mu}\sum_{n=1}^{\infty}\mu\frac{\mu^{n-1}}{(n-1)!} = \mu e^{-\mu}\sum_{n=1}^{\infty}\frac{\mu^{n-1}}{(n-1)!} = \mu e^{-\mu}e^{\mu} = \mu = 2.5$

TH A2

First law, easy

In the adiabatic expansion, into vacuum the energy of the gas is constant. This is a free expansion, so zero work is done by the gas.

Thus:

$$E_2 = E_1$$

$$\frac{3}{2}RT_2 - \frac{a}{V_2} = \frac{3}{2}RT_1 - \frac{a}{V_1} \Rightarrow \underline{T_2 = T_1 + \frac{2}{3R}\left(\frac{1}{V_2} - \frac{1}{V_1}\right)}$$

TH A3

Maxwell relations, easy

We use Helmholtz free energy $F \equiv U - TS$

$$dF = d(U-TS) = dU - TdS - SdT$$

recall that $dU = TdS - p\,dv$

then

$$dF = \cancel{TdS} - p\,dv - \cancel{TdS} - SdT = -p\,dv - SdT$$

The right-hand side is a full differential, thus

$$\underline{\left(\frac{\partial P}{\partial T}\right)_S = \left(\frac{\partial S}{\partial v}\right)_T}$$

Used: $dF = \left(\frac{\partial F}{\partial x}\right)_y dx + \left(\frac{\partial F}{\partial y}\right)_x dy \equiv X\,dx + Y\,dy$

$$\frac{\partial^2 F}{\partial x \partial y} = \left(\frac{\partial X}{\partial y}\right)_x = \left(\frac{\partial Y}{\partial x}\right)_y$$

Second law and heat engines, easy TH A4

The magnitude of the work done by the engine equals to the area enclosed by the cycle in the PV diagram given:

$$W = \frac{1}{2}\left(P_{high} - P_{low}\right)\left(V_{high} - V_{low}\right)$$

$$W = \frac{1}{2} \cdot 10^6 \cdot 0.020 = 10\,000 \text{ J}$$

The efficiency is

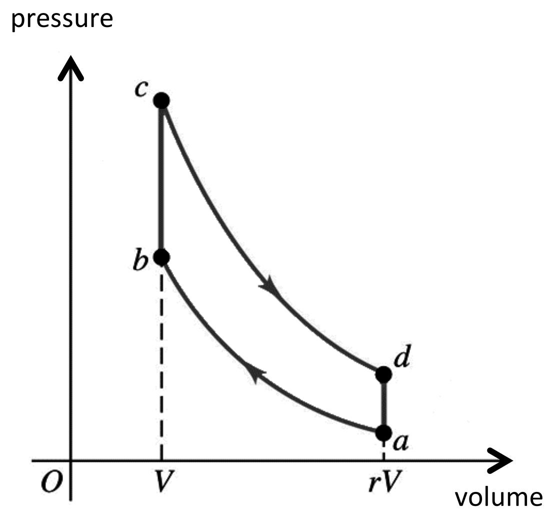
$$\eta = \frac{W}{Q_{absorbed}} = \frac{10\,000 \text{ J}}{50\,000 \text{ J}} = 0.20$$

TH B1

The pV diagram shows the *Otto cycle*, which is an idealized model of the thermodynamic processes in a gasoline engine. Processes $a \to b$ and $c \to d$ are adiabatic, and processes $b \to c$ and $d \to a$ are isochoric (taking place at constant volume). The quantity $r > 1$ is called the compression ratio. Show that the efficiency η of the Otto cycle is

$$\eta = 1 - \frac{1}{r^{\gamma-1}},$$

where $\gamma = C_p / C_V$.



ANSWER

$$Q^{a\to b} = 0 \qquad Q^{b\to c} = nC_V(T_c - T_b) > 0$$

$$Q^{c\to d} = 0 \qquad Q^{d\to a} = nC_V(T_a - T_d) < 0$$

$$\eta = \frac{W}{Q^{b\to c}} = \frac{Q^{b\to c} + Q^{d\to a}}{Q^{b\to c}} = \frac{nC_V(T_c - T_b) + nC_V(T_a - T_d)}{nC_V(T_c - T_b)} = \frac{T_c - T_b + T_a - T_d}{T_c - T_b}$$

$$T_a V_a^{\gamma-1} = T_b V_b^{\gamma-1} \quad \Rightarrow \quad T_a r^{\gamma-1} \cancel{V^{\gamma-1}} = T_b \cancel{V^{\gamma-1}} \quad \Rightarrow \quad T_b = r^{\gamma-1} T_a$$

$$T_c V_c^{\gamma-1} = T_d V_d^{\gamma-1} \quad \Rightarrow \quad T_c \cancel{V^{\gamma-1}} = T_d r^{\gamma-1} \cancel{V^{\gamma-1}} \quad \Rightarrow \quad T_c = r^{\gamma-1} T_d$$

Now $$\eta = \frac{T_c - T_b + T_a - T_d}{T_c - T_d} = \frac{r^{\gamma-1}(T_d - T_a) + T_a - T_d}{r^{\gamma-1}(T_d - T_a)} = \frac{r^{\gamma-1} - 1}{r^{\gamma-1}} = 1 - \frac{1}{r^{\gamma-1}}$$

Probability, hard TH B2

After a sufficiently long time, the bacteria will amount to a huge number $N \gg 10\,000$ without the existence of the predator. That the predator eats bacteria at random, is mathematically equivalent to selecting $n = 10\,000$ bacteria out of N bacteria as survivors. $N \gg n$ means we can assume the probabilities of survival just p and $(1-p)$ for the red and green bacteria, respectively.

We use the binomial distribution to find the probability of m successes (picking the red) out of n trials:

$$\mathbb{P}(m) = C_m^n \left(\frac{1}{2}+p\right)^m \left(\frac{1}{2}-p\right)^{n-m}$$

$$= \frac{n!}{m!(n-m)!} \left(\frac{1}{2}+p\right)^m \left(\frac{1}{2}-p\right)^m$$

First law, hard TH B3

According to the first law,

$$dU = dQ - dW \qquad dQ \to TdS$$

$$TdS = dU + dW \qquad dW \to pdV$$

$$dS = \frac{dU + pdV}{T} = \left[\frac{1}{T}\left(\frac{\partial U}{\partial V}\right)_T + \frac{p}{T}\right]dV + \frac{1}{T}\left(\frac{\partial U}{\partial T}\right)_V dT$$

where we used $dU = \left(\frac{\partial U}{\partial V}\right)_T dV + \left(\frac{\partial U}{\partial T}\right)_V dT$.

Next, substitute given $p(T,V)$ and $u(T,V)$ of the new matter:

$$dS = \frac{BT^{n-1} + AT^2}{V} dV + \left(\frac{f'(T)}{T} + nBT^{n-2} \ln \frac{V}{V_0}\right) dT$$

From the condition that dS is complete differential:

$$\frac{\partial}{\partial T}\left(\frac{BT^{n-1} + AT^2}{V}\right) = \frac{\partial}{\partial V}\left(\frac{f'(T)}{T} + nBT^{n-2} \ln \frac{V}{V_0}\right)$$

we get $2AT - BT^{n-2} = 0$ for any T $\Rightarrow \boxed{n = 3,\ B = 2A}$

Free energies, hard. TH B4

For an ideal gas, we have $dU = nC_v dT$, and $U = nC_vT + U_0$, where U_0 is the internal energy of the system when $T=0$.

$$TdS = dU + p\,dV$$

$$dS = \frac{nC_v}{T}dT + \frac{p}{T}dV.$$ Integrating this, we get

$$S = \frac{3nR}{2}\ln T + nR\ln V + \xi,$$ where ξ is a constant

Assuming $S = S_0$ when $T = T_0$, $V = V_0$, we get

$$S = \frac{3nR}{2}\ln\frac{T}{T_0} + nR\ln\frac{V}{V_0} + S_0$$

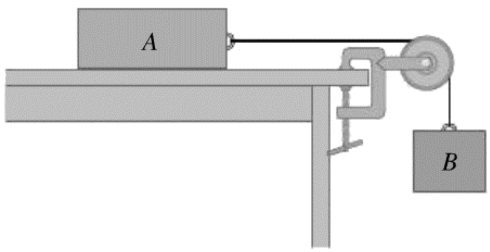
Then

$$F \equiv U - TS = \frac{3nRT}{2} - \left(\frac{3nRT}{2}\ln\frac{T}{T_0} - nRT\ln\frac{V}{V_0}\right) + F_0$$

where $F_0 = U_0 - T_0S_0$

MC A1

Mechanics easy



Two block are connected by a massless cord which rolls over a massless and frictionless pulley. Block A sits on a level, frictionless surface, and is sliding to the right. The masses of the blocks A and B are $m_{\text{A}} = 7\text{ kg}$ and $m_{\text{A}} = 3\text{ kg}$. Calculate the tension T in the cord.

<u>SOLUTION</u>

The blocks have the same acceleration, so

$$\frac{T}{m_{\text{A}}} = a_{\text{A}} = a_{\text{B}} = \frac{m_{\text{B}}g - T}{m_{\text{B}}} \quad \Rightarrow \quad m_{\text{B}}T = m_{\text{A}}m_{\text{B}}g - m_{\text{A}}T \quad \Rightarrow \quad (m_{\text{A}} + m_{\text{B}})T = m_{\text{A}}m_{\text{B}}g \quad \Rightarrow$$

$$T = \frac{m_{\text{A}}m_{\text{B}}}{m_{\text{A}} + m_{\text{B}}}g = \frac{7 \times 3}{7 + 3}g = \frac{21}{10}g = 2.1g = 20.6\text{ N}$$

Check:

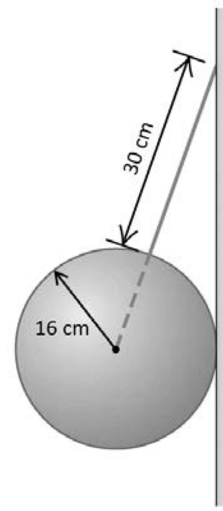
$$a_{\text{A}} = \frac{F_{\text{A}}}{m_{\text{A}}} = \frac{T}{m_{\text{A}}} = \frac{m_{\text{B}}}{m_{\text{A}} + m_{\text{B}}}g$$

$$a_{\text{B}} = \frac{F_{\text{B}}}{m_{\text{B}}} = \frac{m_{\text{B}}g - T}{m_{\text{B}}} = g - \frac{m_{\text{A}}}{m_{\text{A}} + m_{\text{B}}}g = \frac{m_{\text{A}} + m_{\text{B}}}{m_{\text{A}} + m_{\text{B}}}g - \frac{m_{\text{A}}}{m_{\text{A}} + m_{\text{B}}}g = \frac{m_{\text{B}}}{m_{\text{A}} + m_{\text{B}}}g = a_{\text{A}} \quad \text{OK}$$

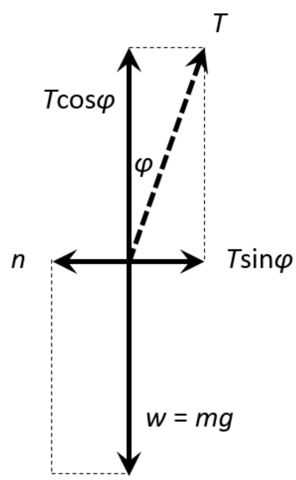
MC A2

Mechanics easy

A solid, uniform, 45 kg ball of radius 16 cm is supported against a vertical, frictionless wall by a thin, 30 cm wire of negligible mass. Find the tension in the wire and the magnitude of the force that the ball exerts on the wall.



ANSWER



$$n = T\sin\varphi \quad \text{and} \quad mg = T\cos\varphi$$

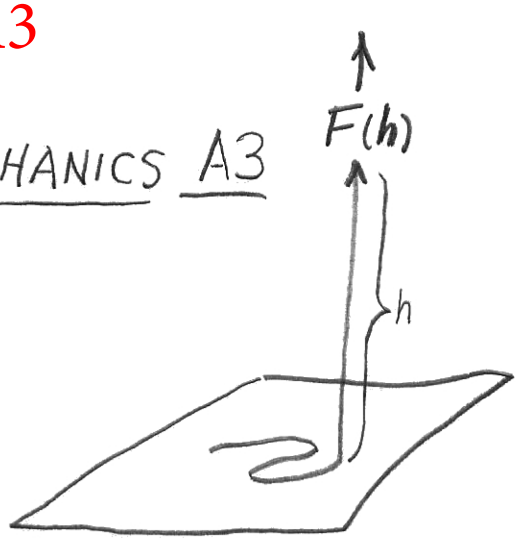
$$\sin\varphi = \frac{16}{30+16} \quad \Rightarrow \quad \varphi = 20.354°$$

$$\frac{n}{mg} = \tan\varphi \quad \Rightarrow \quad n = mg\tan\varphi = 164 \text{ N}$$

$$T = \frac{mg}{\cos\varphi} = 470 \text{ N}$$

MC A3

MECHANICS A3



$$F_{TOT,y} = \dot{p}$$
$$= m\dot{v} + \dot{m}v$$

$$F(h) - m(h)g$$
$$= \dot{m}v$$

$$F(h) = \mu h g + \mu v^2$$
$$= \mu(hg + v^2)$$

MC A4

(a) We note that the angle between the cable and the strut is

$$\alpha = \theta - \phi = 45^\circ - 30^\circ = 15^\circ.$$

The angle between the strut and any vertical force (like the weights in the problem) is $\beta = 90^\circ - 45^\circ = 45^\circ$. Denoting $M = 225$ kg and $m = 45.0$ kg, and ℓ as the length of the boom, we compute torques about the hinge and find

$$T = \frac{Mg\ell\sin\beta + mg\left(\frac{\ell}{2}\right)\sin\beta}{\ell\sin\alpha} = \frac{Mg\sin\beta + mg\sin\beta/2}{\sin\alpha}.$$

The unknown length ℓ cancels out and we obtain $T = 6.63 \times 10^3$ N.

(b) Since the cable is at 30° from horizontal, then horizontal equilibrium of forces requires that the horizontal hinge force be

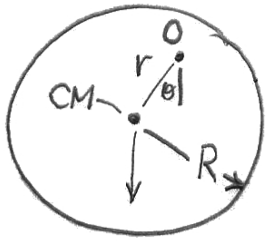
$$F_x = T\cos 30^\circ = 5.74 \times 10^3\,\text{N}.$$

and vertical equilibrium of forces gives the vertical hinge force component:

$$F_y = Mg + mg + T\sin 30^\circ = 5.96 \times 10^3\,\text{N}.$$

MC B1

MECHANICS B1



$$\tau = I\alpha$$

$$-Mgr\sin\theta = I_o\ddot{\theta}$$

$$-Mgr\theta \simeq \left(\tfrac{1}{2}MR^2 + Mr^2\right)\ddot{\theta}$$

$$-gr\theta \simeq \left(\tfrac{1}{2}R^2 + r^2\right)\ddot{\theta}$$

This is analogous to

$$-kx = m\ddot{x}$$

$$\omega = \sqrt{\frac{k}{m}} = \frac{2\pi}{T}$$

$T = 2\pi\sqrt{\frac{m}{k}}$ or, by analogy

$$T = 2\pi\sqrt{\frac{\frac{1}{2}R^2 + r^2}{gr}}$$

to maximize T, we should set

$$\frac{d}{dr}\left(\frac{R^2}{2r} + r\right) = 0 = 1 - \frac{R^2}{2r^2}$$

$$2r^2 = R^2 \; ; \quad \boxed{r = \frac{1}{\sqrt{2}}R}$$

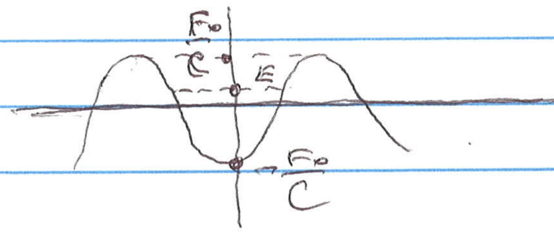
MC B2

Mech 1 Solution

(a) $F(x) = -F_0 \sin(cx)$

$$V(x) = -\int F(x)\,dx = -\frac{F_0}{c}\cos(cx)$$

(choose const $= 0$).



(b) $$\frac{mv_0^2}{2} - \frac{F_0}{C} = \frac{mv^2}{2} - \frac{F_0}{c}\cos(cx)$$

$$v = \sqrt{v_0^2 - \frac{2F_0}{mc}(1-\cos cx)}$$

(c) $$E = \frac{mv_0^2}{2} - \frac{F}{C} < \frac{F_0}{C}$$ (for bounded motion)

$$v_0 < 2\left(\frac{F_0}{mc}\right)^{1/2}$$

(d) $F \approx -F_0 cx \equiv -kx$

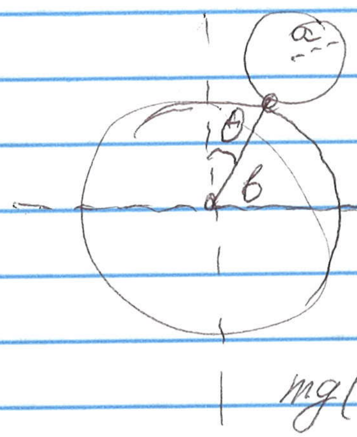
$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{F_0 C}{m}}$$

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{F_0 C}}$$

$$\text{amplitude} = \frac{v_0}{\omega_0} = v_0\sqrt{\frac{m}{F_0 C}}$$

MC B3

Mech hard 2 Solution



Let us count the pot energy of the sphere from the level corresponding to the center of the cylinder. Then

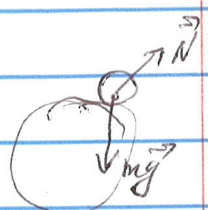
$$mg(b+a) = mg(b+a)\cos\theta + \frac{mv^2}{2} + \frac{I\omega^2}{2}$$

For a rough surface there is no slipping and

$$\omega = \frac{v}{a}$$

$$v^2 = \frac{2mg(b+a)(1-\cos\theta)}{m+\frac{I}{a^2}}$$

Newton eq. for the motion of the center of mass of the sphere



$$\frac{mv^2}{b+a} = mg\cos\theta - N$$

Sphere leaves when $N=0 \rightarrow v^2 = (b+a)g\cos\theta$

$$\frac{2g(b+a)(1-\cos\theta)}{1+\frac{I}{ma^2}} = (b+a)g\cos\theta$$

$$\cos\theta = \frac{2}{3+\frac{I}{ma^2}}$$

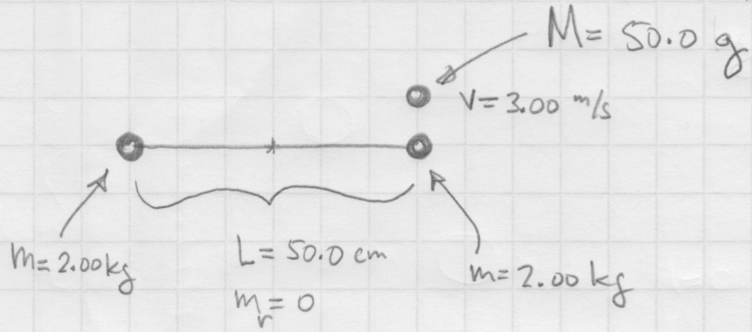
For a uniform sphere $I = \frac{2}{5}ma^2$ and $\cos\theta = \frac{2}{3+\frac{2}{5}} = \frac{10}{17}$

for a hollow sphere $I = \frac{2}{3}ma^2$ $\cos\theta = \frac{2}{3+\frac{2}{3}} = \frac{6}{11}$

$\frac{6}{11} < \frac{10}{17}$ $\quad\theta_{hollow} > \theta_{uniform}$

MC B4

Section 1-1
Johnson



(a) Conserve angular momentum: $L_i = L_f$, $L = I\omega$

$$L_i = I_p \omega_p = \left[M\left(\frac{L}{2}\right)^2\right]\left[\frac{V}{(L/2)}\right] = M\frac{L}{2}V$$

$$L_f = (I_p + I)\,\omega = \left[M\left(\frac{L}{2}\right)^2 + 2m\left(\frac{L}{2}\right)^2\right]\omega = (M+2m)\left(\frac{L}{2}\right)^2\omega$$

$$M\frac{L}{2}V = (M+2m)\left(\frac{L}{2}\right)^2\omega$$

solve for ω

$$\omega = \frac{M}{(M+2m)}\frac{2}{L}V = \frac{50.0\times10^{-3}\,kg}{(50.0\times10^{-3}\,kg + 2\cdot2.00\,kg)}\frac{2(3.00\,m/s)}{(50.0\times10^{-2}\,m)}$$

$$\boxed{\omega = 0.148\ \frac{rad}{sec}}$$

(b) $K_i = \frac{1}{2}Mv^2$

$K_f = \frac{1}{2}(I_p + I)\,\omega^2$

$$\frac{K_f}{K_i} = \frac{(I_p+I)\,\omega^2}{Mv^2} = \frac{(M+2m)\left(\frac{L}{2}\right)^2\omega^2}{Mv^2}$$

Section 1-1
Johnson

$$\frac{K_f}{K_i} = \frac{(50.0\times10^{-3}\,kg + 2(2.00\,kg))}{(50.0\times10^{-3}\,kg)(3.00\,m/s)^2}\left(\frac{50.0\times10^{-2}\,m}{2}\right)^2\left(0.148\,\frac{rad}{s}\right)^2$$

$$\boxed{\frac{K_f}{K_i} = 1.23\times10^{-2}}$$

(c) after the collision, energy is conserved

$$K_i = \alpha\,\tfrac{1}{2}Mv^2,\quad \alpha = 1.23\times10^{-2} \text{ (from part b)}$$

$$U_i = 0$$

$$K_f = 0$$

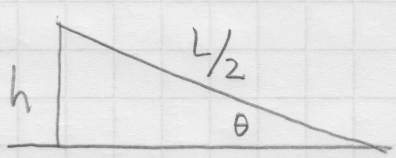
$$U_f = Mgh + \underbrace{mgh - mgh}$$

Note: h must be positive, so the system rotates through $180° + \theta$

the two balls cancel each other because one is above the zero point and the other is below

$$\alpha\,\tfrac{1}{2}Mv^2 = Mgh \rightarrow h = \frac{\alpha v^2}{2g} = \frac{L}{2}\sin\theta$$

$$\sin\theta = \frac{\alpha v^2}{Lg},\quad \theta = \sin^{-1}\left[\frac{\alpha v^2}{Lg}\right]$$



$$\sin\theta = \frac{h}{L/2},\quad h = \frac{L}{2}\sin\theta$$

$$\theta = 1.3°$$

$$\boxed{\text{system rotates } 181°}$$