Thermo A1

Suppose 10.0 g of water at a temperature of 100.0 °C is in an insulated cylinder equipped with a piston that maintains a constant pressure of p=101.3 kPa. Enough hat is added to the water to vaporize it to steam at a temperature of 100.0 °C. The volume of the water is $V_{water} = 10.0$ cm^3, and the volume of the steam is $V_{steam} = 16900$ cm^3. What is the change in internal energy of the water?

The process is carried out at a constant pressure. The work done is computed as follows:

$$W = \int_{V_1}^{V_2} p\,dV = p\int_{V_i}^{V_f} dV = P(V_{steam} - V_{water})$$

$$W = (101.3 \cdot 10^3 \text{ Pa})(16900 \cdot 10^{-6} \text{ m}^3 - 10.0 \cdot 10^{-6} \text{ m}^3) = 1710 \text{ J}$$

From the First Law:

$$\Delta U = Q - W$$

Thermal energy transferred is the heat to vaporize the water:

$$Q = m L_v = (10.0 \cdot 10^{-3} \text{ kg})(2.26 \cdot 10^{6} \text{ J/kg}) = 22\,600 \text{ J}$$

Then

$$\Delta U = 22600 \text{ J} - 1710 \text{ J} = \underline{20\,900 \text{ J}}$$

Thermo A2

A Carnot engine takes 3000 J of heat from a thermal reservoir that has a temperature of $T_H = 500$ K and discards heat to a thermal reservoir with a temperature $T_H = 325$ K. How much work does the Carnot engine do in this process?

Efficiency for a heat engine is

$$\varepsilon = \frac{W}{Q_H}$$

for a Carnot engine this becomes (formula sheet)

$$\varepsilon = 1 - \frac{T_L}{T_H}$$

We can combine the two:

$$\frac{W}{Q_H} = 1 - \frac{T_L}{T_H}$$

and find work done:

$$W = Q_H\left(1 - \frac{T_L}{T_H}\right) = 3000\ \text{J} \cdot \left(1 - \frac{325\ \text{K}}{500\ \text{K}}\right)$$

$$= \underline{1050\ \text{J}}$$

Thermo A3

Suppose we start with 2.00 kg of water at a temperature of 20.0 °C and warm the water until it reaches a temperature of 80.0 °C. What is the change in entropy of the water?

The change in entropy is related to the flow of the heat as

$$\Delta S = \int_{T_i}^{T_f} \frac{dQ}{T}$$

The heat Q required to raise the temperature of a mass m of water is given by

$$Q = cm\Delta T$$

and for a small temperature change

$$dQ = cm\,dT$$

Therefore

$$\Delta S = \int_{T_i}^{T_f} \frac{cm\,dT}{T} = cm\int_{T_i}^{T_f} \frac{dT}{T} = cm \ln \frac{T_f}{T_i}$$

$$T_i = 293\ K$$

$$T_f = 353\ K$$

$$\Delta S = 4.19 \frac{kJ}{kg\,K} \cdot 2.00\,kg \ln \frac{353\ K}{293\ K} \approx \underline{1.56 \frac{kJ}{K}}$$

Thermo A4

$$T_2 = T_1\left(\frac{V_1}{V_2}\right)^{\gamma-1} = (300\ \mathrm{K})(15.0)^{0.40} = 886\ \mathrm{K} = 613^\circ\mathrm{C}$$

$$p_2 = p_1\left(\frac{V_1}{V_2}\right)^{\gamma} = (1.01 \times 10^5\ \mathrm{Pa})(15.0)^{1.40}$$

$$= 44.8 \times 10^5\ \mathrm{Pa} = 44\ \mathrm{atm}$$

The work done is

$$W = \frac{1}{\gamma - 1}(p_1V_1 - p_2V_2)$$

Using $V_1/V_2 = 15.0$, we have

$$W = \frac{1}{1.400 - 1}\left[\begin{array}{l}(1.01 \times 10^5\ \mathrm{Pa})(1.00 \times 10^{-3}\ \mathrm{m}^3) \\ -\ (44.8 \times 10^5\ \mathrm{Pa})\left(\dfrac{1.00 \times 10^{-3}\ \mathrm{m}^3}{15.0}\right)\end{array}\right]$$

$$= -494\ \mathrm{J}$$

Thermo B1

I show you four unusual six-sided dice. They are unusual, because they do not have the numbers 1 through 6 on their six faces. Instead, here's what the dice look like (what I'm going to do is list the six numbers on the faces, in increasing order):

- Dice A: 0, 0, 4, 4, 4, 4
- Dice B: 3, 3, 3, 3, 3, 3
- Dice C: 2, 2, 2, 2, 6, 6
- Dice D: 1, 1, 1, 5, 5, 5

I play the following game with you: you choose one of the dice, then I choose another. Having made our choices, we both roll our respective dice. You win the game if you roll a higher number than me (notice that ties are impossible).
Which dice do you choose, in order to maximize your probability of winning this game, and what is your probability to win with that choice?

Consider all options. Denote players P1, P2.

1) First dice is A, second player picks D.
if P2 rolls 5, he always wins → $\frac{1}{2}$
if P2 rolls 1, he wins $\frac{1}{3}$ of the time
P2 prob to win is $\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{3}=\frac{2}{3}$

2) P1 picks B, second player has to pick A.
Obviously, is P2 rolls 4 ($\frac{2}{3}$ prob), he wins, otherwise he looses.
P2 prob to win is $\frac{2}{3}$

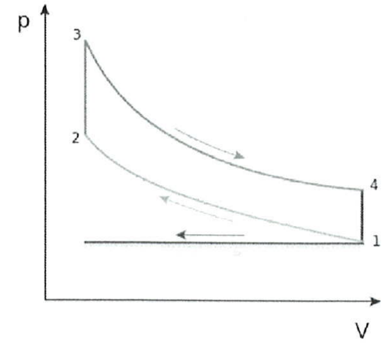
3) P1 picks C, P2 has to pick B.
P2 outcomes are all the same, for P1 rolling "2"
P2 ~~looses~~ wins ($\frac{2}{3}$ of the time), otherwise P2 looses.
P2 prob to win is $\frac{2}{3}$

4) P1 picks D, P2 has to pick C.
If P2 rolls "2", he wins half of the time, otherwise he always wins.
P2 prob to win is $\frac{2}{3}\cdot\frac{1}{2}+\frac{1}{3}\cdot 1=\frac{2}{3}$

– Probability continued –

No matter what player P1 chooses, the out come does not change. With the optimal choice by P2, player P1 wins in 1/3 of the times regardless of the P1's dice.

Thermo B2



The Otto cycle, used in modern internal combustion engines, consists of two adiabatic processes and two constant-volume processes. Given that the volume of the fuel-air mixture at points 1 and 4 is V_1 and the volume of the mixture at points 2 and 3 is V_2, find the efficiency of the Otto cycle (i.e. net work done over the heat gained per cycle).

$1 \to 2$: adiabatic process. $Q_{12} = 0$
$2 \to 3$: isohoric, $W_{23} = 0$
$3 \to 4$: adiabatic, $Q_{34} = 0$
$4 \to 1$: isohoric, $W_{41} = 0$

The efficiency of the cycle is

$$\varepsilon = \frac{W}{Q_{gain}} = \frac{Q_{gain} - Q_{lost}}{Q_{gain}}.$$

Heat is gained in the $2 \to 3$ segment, at $V = const$

$Q_{gain} = n C_V (T_3 - T_2)$, where C_V is the molar heat capacity.

Heat is lost in the $4 \to 1$ segment:

$$Q_{lost} = n C_V (T_4 - T_1)$$

Then

$$\varepsilon = \frac{Q_{gain} - Q_{lost}}{Q_{gain}} = \frac{n C_V (T_3 - T_2) - n C_V (T_4 - T_1)}{n C_V (T_3 - T_2)}$$

$$= 1 - \frac{T_4 - T_1}{T_3 - T_2}.$$ We need to convert this to V_1, V_2

For an adiabatic process,

$PV^{\gamma} = const$ and $TV^{\gamma - 1} = const$

– Otto cycle continues –

Thus

$$T_4 V_1^{\gamma-1} = T_3 V_2^{\gamma-1} \quad (1)$$

$$T_1 V_1^{\gamma-1} = T_2 V_2^{\gamma-1} \quad (2)$$

Subtract (1) – (2):

$$(T_4 - T_1) V_1^{\gamma-1} = (T_3 - T_2) V_2^{\gamma-1} \Rightarrow \frac{T_4 - T_1}{T_3 - T_2} = \left(\frac{V_2}{V_1}\right)^{\gamma-1}$$

Finally

$$\boxed{\varepsilon = 1 - \left(\frac{V_2}{V_1}\right)^{\gamma-1}}$$

$$\left(\text{or, } \varepsilon = 1 - \left(\frac{1}{r}\right)^{\gamma-1} = 1 - r^{1-\gamma} \right.$$

$$\left. \text{where } r = \frac{V_1}{V_2} \right)$$

Thermo B3

For the internal energy $U(V,T)$ compute $(\partial U/\partial V)_P$ through observable quantities only: pressure P, temperature T, volume V, isothermal compressibility κ, and thermal expansion coefficient α (refer to the formula sheet for the definitions of κ and α).

$U(T,V) \Rightarrow dU = \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV$

divide by dV at $p = \text{const}$

$$\left(\frac{\partial U}{\partial V}\right)_p = \underbrace{\left(\frac{\partial U}{\partial T}\right)_V}_{C_V}\left(\frac{\partial T}{\partial V}\right)_p + \left(\frac{\partial U}{\partial V}\right)_T$$

$$= C_V\left(\frac{\partial T}{\partial V}\right)_p + \left(\frac{\partial U}{\partial V}\right)_T$$

$\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P$ from formula sheet

$$\Rightarrow \left(\frac{\partial T}{\partial V}\right)_P = \frac{1}{\alpha V}$$

Now $\left(\frac{\partial U}{\partial V}\right)_P = \frac{C_V}{\alpha V} + \left(\frac{\partial U}{\partial V}\right)_T$

Next: $dU = TdS - PdV$ (formula sheet)

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial S}{\partial V}\right)_T - P$$

Use Maxwell's relation $\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V$

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P$$

— entropy continued —

Use $\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial y}{\partial z}\right)_x = -1$ for P, V, T:

$$\left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial V}{\partial P}\right)_T \left(\frac{\partial T}{\partial V}\right)_P = -1$$

$$\left(\frac{\partial P}{\partial t}\right)_V = -\frac{(\partial V/\partial T)_P}{(\partial V/\partial P)_T} = -\frac{\alpha V}{-\kappa V} = \frac{\alpha}{\kappa}$$

Finally, substituting the pieces.

$$\left(\frac{\partial U}{\partial V}\right)_P = \frac{C_V}{\alpha V} + \left(\frac{\partial U}{\partial V}\right)_T = \frac{C_V}{\alpha V} + T\left(\frac{\partial P}{\partial t}\right)_V - P$$

$$= \frac{C_V}{\alpha V} + \frac{T\alpha}{\kappa} - P$$

Solution Thermo B4

Solution: a) If all spins point up we get

$E/N = J$

b) The partition function: $Z = Z_1^N = e^{\frac{J \cdot N}{T}}\left(1 + e^{-\frac{2JN}{T}}\right)^N$

Free energy: $F = -T \ln Z = -NT \ln\left(1 + e^{-\frac{2JN}{T}}\right) - NJ$

The energy: $E = F - T \cdot \frac{\partial F}{\partial T} = N T^2 \frac{\partial}{\partial T} \ln Z_1 - N J$

$$J + \frac{E}{N} = T^2 \frac{\partial}{\partial T} \ln\left(1 + e^{-\frac{2J}{T}}\right) = 2J \frac{e^{-\frac{2J}{T}}}{1 + e^{-\frac{2J}{T}}} = 2J \frac{x}{1+x}$$

$$x = e^{-\frac{2J}{T}}$$

Clearly: $\max\left(\frac{E}{N}\right) = 0$

Mech A1

$K_1 = K_2 = 0$

$U_1 = U_{1,\text{el}} = \frac{1}{2}kx_1^2 = \frac{1}{2}(100\ \text{N/m})(0.200\ \text{m})^2 = 2.00\ \text{J}$

$U_2 = U_{2,\text{el}} = 0$, since after the block leaves the spring has given up all its stored energy

$W_{\text{other}} = W_f = (f_\text{k} \cos\phi)s = \mu_\text{k} mg\left(\cos\phi\right)s = -\mu_\text{k} mgs$, since $\phi = 180°$ (The friction force is directed opposite to the displacement and does negative work.)

Putting all this into $K_1 + U_1 + W_{\text{other}} = K_2 + U_2$ gives

$U_{1,\text{el}} + W_f = 0$

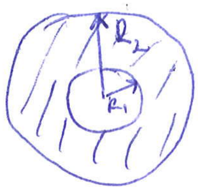
$\mu_\text{k} mgs = U_{1,\text{el}}$

$$\mu_\text{k} = \frac{U_{1,\text{el}}}{mgs} = \frac{2.00\ \text{J}}{(0.50\ \text{kg})(9.80\ \text{m/s}^2)(1.00\ \text{m})} = 0.41.$$

A2 A ball of mass 1 kg is falling vertically down. Assuming the quadratic air resistance, $F = cv^2$ (v is the ball's speed) with $c = 0.01$ kg/m,

(a) find the terminal speed;

(b) Assuming that the ball is moving with the terminal speed, find the energy dissipated per second.



A3 A spherical shell filled with a fluid of mass M has the inner radius R_1 and the outer radius R_2. A particle of mass m is placed at the distance r from the center. Find the force on the particle at (a) $r < R_1$, (b) $R_1 < r < R_2$ and (c) $r > R_2$. and sketch $F(r)$

Answers:

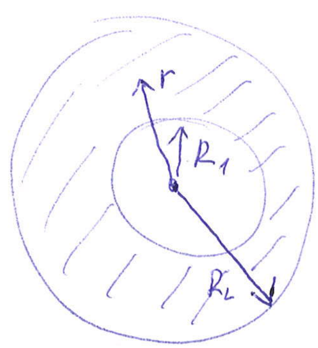
KEY A2

(a) $mg = c v_t^2$

$$v_t = \sqrt{\frac{mg}{c}} = \sqrt{\frac{1\,kg \cdot 9.8\,\frac{m}{s^2}}{0.01\,\frac{kg}{m}}} = 31.3\,\frac{m}{s}$$

(b) $$W = \frac{d}{dt}(mgh) = mgv = 1\,kg \cdot 9.8\,\frac{m}{s^2} \cdot 31.3\,\frac{m}{s} = 306.7\,\frac{J}{s}$$

KEY A3



(1) $r < R_1$ $F = 0$

(2) $R_1 < r < R_2$

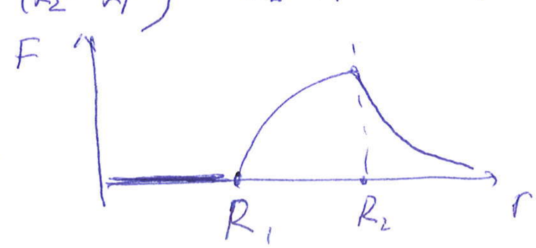
$$F = G\frac{M'm}{r^2}, \quad M' = \frac{4}{3}\pi\rho(r^3 - R_1^3)$$

ρ from $M = \frac{4}{3}\pi\rho(R_2^3 - R_1^3)$

therefore $$M' = M\frac{r^3 - R_1^3}{R_2^3 - R_1^3}$$

$$F = GMm\frac{r^3 - R_1^3}{r^2(R_2^3 - R_1^3)} = \frac{GMm}{R_2^3 - R_1^3}\left(r - \frac{R_1^3}{r^2}\right)$$

(3) $R > R_2$: $F = \frac{GMm}{r^2}$



Mech A4

Solution. If long after the collision both masses move in the same direction, it means $m_1 > m_2$.

$$\begin{cases} \dfrac{m_1 V_0^2}{2} = \dfrac{m_1 V_1^2}{2} + \dfrac{m_2 V_2^2}{2} \\ m_1 V_0 = m_1 V_1 + m_2 V_2 \end{cases}$$

$$\Downarrow$$

$$V_1 = \frac{m_1 - m_2}{m_1 + m_2} V_0$$

$$V_2 = \frac{2 m_1}{m_1 + m_2} V_0$$

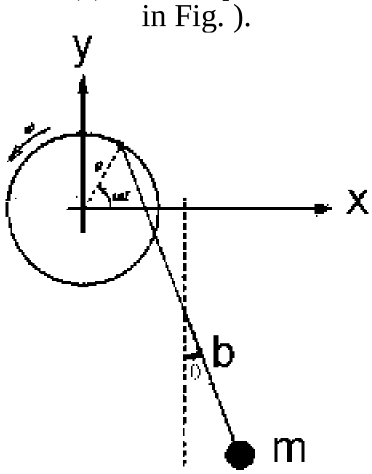
B1

The point of support of a simple pendulum of length *b* moves on a massless rim of radius *a* rotating with constant angular velocity ω.

(a) Obtain the expression for the Cartesian components of the velocity of the mass m.

(b) The Lagrangian

(c) The equation of motion for θ (obtain the angular acceleration for the angle θ shown in Fig.).



The point of support of a simple pendulum of length *b* moves on a massless rim of radius *a* rotating with constant angular velocity ω. Obtain the expression for the Cartesian components of the velocity and acceleration of the mass *m*. Obtain also the angular acceleration for the angle θ shown in Figure

We choose the origin of our coordinate system to be at the center of the rotating rim.
The Cartesian components of mass m become

$$\left.\begin{aligned} x &= a\cos\omega t + b\sin\theta \\ y &= a\sin\omega t - b\cos\theta \end{aligned}\right\}$$

The velocities are

$$\left.\begin{aligned} \dot{x} &= -a\omega\sin\omega t + b\dot{\theta}\cos\theta \\ \dot{y} &= a\omega\cos\omega t + b\dot{\theta}\sin\theta \end{aligned}\right\}$$

the acceleration:

$$\ddot{x} = -a\omega^2\cos\omega t + b(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)$$

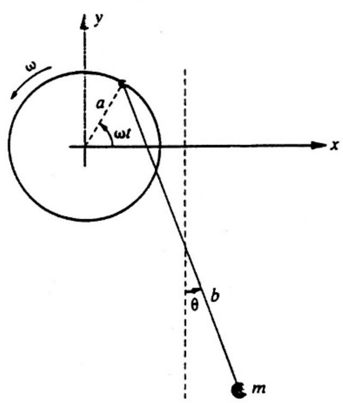
$$\ddot{y} = -a\omega^2\sin\omega t + b(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)$$

the single generalized coordinate is θ.

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$U = mgy$$

where $U = 0$ at $y = 0$.



$$L = T - U = \frac{m}{2}[a^2\omega^2 + b^2\dot{\theta}^2 + 2b\dot{\theta}a\omega\sin(\theta - \omega t)]$$
$$-mg(a\sin\omega t - b\cos\theta)$$

$$L = T - U = \frac{m}{2}[a^2\omega^2 + b^2\dot{\theta}^2 + 2b\dot{\theta}a\omega \sin(\theta - \omega t)] - mg(a \sin \omega t - b \cos \theta)$$

the Lagrange equation of motion for θ are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = mb^2\ddot{\theta} + mba\omega(\dot{\theta} - \omega)\cos(\theta - \omega t)$$

$$\frac{\partial L}{\partial \theta} = mb\dot{\theta}a\omega \cos(\theta - \omega t) - mgb \sin \theta$$

the equation of motion $\quad \ddot{\theta} = \frac{\omega^2 a}{b}\cos(\theta - \omega t) - \frac{g}{b}\sin\theta$

Notice that this result reduces to the well-known equation of motion for a simple pendulum if $\omega = 0$.

B2

Find the force law for a central-force field that allows a particle to move in a logarithmic spiral orbit given by $r = ke^{\alpha\theta}$, where k and α are constants.

Note: the equation of the orbit

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2u^2}F(\frac{1}{u})$$

$$u = \frac{1}{r}, \quad L = mr^2\dot{\theta} = const$$

Find the force law for a central-force field that allows a particle to move in a logarithmic spiral orbit given by $r = ke^{\alpha\theta}$, where k and α are constants.

Solution: We use Equation 8.21 to determine the force law $F(r)$. First, we determine

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{d}{d\theta}\left(\frac{e^{-\alpha\theta}}{k}\right) = \frac{-\alpha e^{-\alpha\theta}}{k}$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{\alpha^2 e^{-\alpha\theta}}{k} = \frac{\alpha^2}{r}$$

From Equation 8.21, we now determine $F(r)$.

$$F(r) = \frac{-l^2}{\mu r^2}\left(\frac{\alpha^2}{r} + \frac{1}{r}\right)$$

$$F(r) = \frac{-l^2}{\mu r^3}(\alpha^2 + 1) \qquad \textbf{(8.22)}$$

Thus, the force law is an attractive inverse cube.

Determine $r(t)$ and $\theta(t)$

Solution: From Equation 8.10, we find

$$\dot{\theta} = \frac{l}{\mu r^2} = \frac{l}{\mu k^2 e^{2\alpha\theta}} \qquad \textbf{(8.23)}$$

Rearranging Equation 8.23 gives

$$e^{2\alpha\theta}\, d\theta = \frac{l}{\mu k^2}\, dt$$

and integrating gives

$$\frac{e^{2\alpha\theta}}{2\alpha} = \frac{lt}{\mu k^2} + C'$$

where C' is an integration constant. Multiplying by 2α and letting $C = 2\alpha C'$ gives

$$e^{2\alpha\theta} = \frac{2\alpha lt}{\mu k^2} + C \qquad \textbf{(8.24)}$$

We solve for $\theta(t)$ by taking the natural logarithm of Equation 8.24:

$$\theta(t) = \frac{1}{2\alpha}\ln\left(\frac{2\alpha lt}{\mu k^2} + C\right) \qquad \textbf{(8.25)}$$

We can similarly solve for $r(t)$ by examining Equations 8.23 and 8.24:

$$\frac{r^2}{k^2} = e^{2\alpha\theta} = \frac{2\alpha lt}{\mu k^2} + C$$

$$r(t) = \left[\frac{2\alpha l}{\mu}t + k^2C\right]^{1/2} \qquad \textbf{(8.26)}$$

The integration constant C and angular momentum l needed for Equations 8.25 and 8.26 are determined from the initial conditions.

What is the total energy of the orbit

Solution: The energy is found from Equation 8.14. In particular, we need $\dot{r}$ and $U(r)$.

$$U(r) = -\int F\, dr = \frac{+l^2}{\mu}(\alpha^2 + 1)\int r^{-3}\, dr$$

$$U(r) = -\frac{l^2(\alpha^2 + 1)}{2\mu}\frac{1}{r^2} \qquad \textbf{(8.27)}$$

where we have let $U(\infty) = 0$.

We rewrite Equation 8.10 to determine $\dot{r}$:

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{d\theta}{dr}\frac{dr}{dt} = \frac{l}{\mu r^2}$$

$$\dot{r} = \frac{dr}{d\theta}\frac{l}{\mu r^2} = \alpha k e^{\alpha\theta}\frac{l}{\mu r^2} = \frac{\alpha l}{\mu r} \qquad \textbf{(8.28)}$$

Substituting Equations 8.27 and 8.28 into Equation 8.14 gives

$$E = \frac{1}{2}\mu\left(\frac{\alpha l}{\mu r}\right)^2 + \frac{l^2}{2\mu r^2} - \frac{l^2(\alpha^2 + 1)}{2\mu r^2}$$

$$E = 0 \qquad \textbf{(8.29)}$$

The total energy of the orbit is zero if $U(r = \infty) = 0$.

Mech B3 Solution

Gravitational force on the first satellite:

$$F_1 = G\frac{M_1 M_E}{R^2} \qquad \text{with } M_E = \text{mass of Earth}$$

$$G\frac{M_1 M_E}{R^2} - T = M_1\Omega^2 R \qquad \text{where } T = \text{tension in rope}$$

For the second satellite:

$$G\frac{M_2 M_E}{(R+L)^2} + T = M_2\Omega^2 (R+L)$$

$$G\frac{M_1 M_E}{R^2} - T = M_1\Omega^2 R \quad \Rightarrow \quad \Omega^2 = \frac{1}{M_1 R}\left(G\frac{M_1 M_E}{R^2} - T\right)$$

$$G\frac{M_2 M_E}{(R+L)^2} + T = M_2\Omega^2 (R+L) \quad \Rightarrow \quad \Omega^2 = \frac{1}{M_2 (R+L)}\left(G\frac{M_2 M_E}{(R+L)^2} + T\right)$$

Hence,

$$T = \frac{M_1 M_2}{M_1 R + M_2 (R+L)}\left(\frac{(R+L)^3 - R^3}{R^2 (R+L)^2}\right) M_E G$$

In the limit L << R, and using $g = GM_E / R^2$:

$$T \approx 3\frac{M_1 M_2}{M_1 + M_2}\left(\frac{L}{R}\right) g$$

Mech B4 - Solution.

There will be two contributions to the scale reading. The part that is already on the scale will give Mgx/L. The second part corresponds to the change of momentum of links:

$$\frac{dp}{dt} = v\frac{dm}{dt} = v^2\frac{M}{L} = 2g(x+L)\frac{M}{L}$$

here $v^2 = 2g(x+L)$

The total force: $\dfrac{Mgx}{L} + \dfrac{2g(x+L)M}{L} = 3Mg\dfrac{x}{L} + 2gM$