

QM A1

For PS $a_0 = 2a_B$ since $m = \frac{m_e}{2}$

$$\psi(\vec{r}) = \frac{e^{-r/a_0}}{\pi^{\frac{1}{4}} a_0^{3/2}}$$

$$P(r) = 4\pi r^2 |\psi(\vec{r})|^2 = 4r^2 \frac{e^{-2r/a_0}}{a_0^3} \quad (\text{radial probability density})$$

$$(1) \quad \frac{dP}{dr} \sim e^{-2r/a_0} \left(-\frac{2}{a_0} r^2 + 2r \right) = 0$$

$$r = a_0 = 2a_B$$

$$(2) \quad \langle r \rangle = \int_0^\infty r P(r) dr = \frac{4}{a_0^3} \int_0^\infty r^3 e^{-2r/a_0} dr = \frac{4}{a_0^3} \left(\frac{a_0}{2} \right)^3 !$$
$$= \frac{3}{2} a_0 = 3a_B$$

QM A2

$$\lambda_{\text{dB}} = \frac{h}{p} = \frac{h}{\sqrt{2m_p E}} = 2.03 \times 10^{-11} \text{ m} = 20.3 \text{ pm}$$

$$2d \sin \varphi = n\lambda \Rightarrow d = \frac{1}{2} \frac{n\lambda}{\sin \varphi} = \frac{1}{2} \frac{5\lambda_{\text{dB}}}{\sin 30^\circ} = \\ = \frac{1}{2} \frac{5(2.03 \times 10^{-11})}{\frac{1}{2}} = 1.01 \times 10^{-10} \text{ m} = 0.101 \text{ nm}$$

QM A4

$$L_- Y_{2,-2} = \hbar e^{-i\phi} \left(i \cot \theta \frac{\partial}{\partial \phi} - \frac{\partial}{\partial \theta} \right) \sqrt{\frac{15}{32\pi}} e^{-2i\phi} \sin^2 \theta = \\ = \hbar e^{-i\phi} i \cot \theta \frac{\partial}{\partial \phi} \left(\sqrt{\frac{15}{32\pi}} e^{-2i\phi} \sin^2 \theta \right) - \hbar e^{-i\phi} \frac{\partial}{\partial \theta} \left(\sqrt{\frac{15}{32\pi}} e^{-2i\phi} \sin^2 \theta \right) = \\ = \hbar e^{-i\phi} i \cot \theta \sqrt{\frac{15}{32\pi}} \sin^2 \theta \frac{\partial}{\partial \phi} (e^{-2i\phi}) - \hbar e^{-i\phi} \sqrt{\frac{15}{32\pi}} e^{-2i\phi} \frac{\partial}{\partial \theta} (\sin^2 \theta) = \\ = \hbar e^{-i\phi} i \cot \theta \sqrt{\frac{15}{32\pi}} \sin^2 \theta (-2i) e^{-2i\phi} - \hbar e^{-i\phi} \sqrt{\frac{15}{32\pi}} e^{-2i\phi} 2 \sin \theta \cos \theta = \\ = \hbar e^{-i\phi} \sqrt{\frac{15}{32\pi}} e^{-2i\phi} (i \cot \theta \sin^2 \theta (-2i) - 2 \sin \theta \cos \theta) = \hbar e^{-i\phi} \sqrt{\frac{15}{32\pi}} e^{-2i\phi} \left(2 \frac{\cos \theta}{\sin \theta} \sin^2 \theta - 2 \sin \theta \cos \theta \right) = 0$$

The ladder operator L_- lowers the magnetic quantum number by 1 unit: $L_- |\ell, m\rangle \rightarrow |\ell, m-1\rangle$. But when $\ell = 2, m = -2$ is the lowest possible magnetic quantum number, so it cannot be lowered to $m = -3$, and the result is the null ket $|\emptyset\rangle$, i.e. the ket for which $\langle \Psi | \emptyset \rangle = 0$ for all $|\Psi\rangle$ (and also $\langle \emptyset | \emptyset \rangle = 0$).

QM A3

(a)

$$H\psi = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} A [\cos(kx) + \cos(\frac{kx}{2})]$$
$$= \frac{\hbar^2 A}{2m} [k^2 \cos(kx) + \frac{k^2}{4} \cos(\frac{kx}{2})] \rightarrow n\omega$$

(b) ψ is an even function, yes, parity $P = +1$

(c) $\psi(x, t) = A [\cos(kx) e^{-iE_1 t/\hbar} + \cos(\frac{kx}{2}) e^{-iE_2 t/\hbar}]$
where $E_1 = \frac{\hbar^2 k^2}{2m}$ $E_2 = \frac{\hbar^2 k^2}{8m}$ (from (a)).

(d) p and P are conserved, therefore

$$\langle p \rangle(t) = \underbrace{\langle p \rangle(0)}_0 = 0 \quad P(t) = P(0) = +1$$

$\int \psi p \psi dx = 0$ because $\psi p \psi$ is an odd function.

QM B1

$$\vec{S} = \frac{\hbar}{2} \vec{\Xi}$$

$$(a) \quad \langle S_x \rangle = (\alpha^* \beta^*) \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = (\alpha^* \beta^*) \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \alpha^* \beta + \beta^* \alpha = 2 \operatorname{Re}(\alpha^* \beta)$$

$$\langle S_y \rangle = (\alpha^* \beta^*) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = (\alpha^* \beta^*) \begin{pmatrix} -i\beta \\ i\alpha \end{pmatrix} = -i\alpha^* \beta + i\beta^* \alpha = 2i \operatorname{Im}(\alpha^* \beta)$$

$$\langle S_z \rangle = (\alpha^* \beta^*) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = |\alpha|^2 - |\beta|^2$$

$$(b) \quad \text{we want } \langle S_x \rangle = \langle S_y \rangle = \langle S_z \rangle = 0$$

$$\text{write } \alpha = r e^{i\phi_\alpha}, \beta = r e^{i\phi_\beta}$$

to satisfy the last condition $|\alpha| = |\beta|$

Then from the first two

$$\operatorname{Re}(e^{i(\phi_\beta - \phi_\alpha)}) = 0 \rightarrow \cos(\phi_\beta - \phi_\alpha) = 0 \quad \text{these two are}$$

$$\operatorname{Im}(e^{i(\phi_\beta - \phi_\alpha)}) = 0 \rightarrow \sin(\phi_\beta - \phi_\alpha) = 0 \quad \text{incompatible}$$

therefore $\langle S_x \rangle = \langle S_y \rangle = \langle S_z \rangle = 0$ is impossible

(c)

$$S_x \chi = \pm \frac{\hbar}{2} \chi \quad \chi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$$

$$\text{since } \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\langle S_y \rangle = \frac{1}{2} (1, \pm 1) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} = \frac{1}{2} (1, \pm 1) \begin{pmatrix} \mp i \\ i \end{pmatrix} = 0$$

$$\langle S_z \rangle = \frac{1}{2} (1, \pm 1) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} = \frac{1}{2} (1, \pm 1) \begin{pmatrix} 1 \\ \mp 1 \end{pmatrix} = 0$$

$$S_x^2 + S_y^2 + S_z^2 = \frac{3}{4} \hbar^2$$

$$\text{since } S_x^2 = \frac{\hbar^2}{4} \quad \langle S_y^2 \rangle + \langle S_z^2 \rangle = \frac{\hbar^2}{2} \quad \langle S_y^2 \rangle = \langle S_z^2 \rangle = \frac{\hbar^2}{4}$$

QM B2

$$\hat{H} = \alpha(|\varphi_1\rangle\langle\varphi_2| + |\varphi_2\rangle\langle\varphi_1|)$$

$$(1) \quad \hat{H}^\dagger = \left\{ \alpha(|\varphi_1\rangle\langle\varphi_2| + |\varphi_2\rangle\langle\varphi_1|) \right\}^\dagger = \alpha^* (|\varphi_1\rangle\langle\varphi_2|)^\dagger + \alpha^* (|\varphi_2\rangle\langle\varphi_1|)^\dagger = \alpha(|\varphi_2\rangle\langle\varphi_1|) + \alpha(|\varphi_1\rangle\langle\varphi_2|) = \hat{H}$$

$$\begin{aligned} (2) \quad \hat{H}^2 &= \hat{H}\hat{H} = \alpha(|\varphi_1\rangle\langle\varphi_2| + |\varphi_2\rangle\langle\varphi_1|)\alpha(|\varphi_1\rangle\langle\varphi_2| + |\varphi_2\rangle\langle\varphi_1|) = \\ &= \alpha^2 \left[(|\varphi_1\rangle\langle\varphi_2|)(|\varphi_1\rangle\langle\varphi_2|) + (|\varphi_1\rangle\langle\varphi_2|)(|\varphi_2\rangle\langle\varphi_1|) + (|\varphi_2\rangle\langle\varphi_1|)(|\varphi_1\rangle\langle\varphi_2|) + (|\varphi_2\rangle\langle\varphi_1|)(|\varphi_2\rangle\langle\varphi_1|) \right] = \\ &= \alpha^2 \left[|\varphi_1\rangle\langle\varphi_2| \cancel{|\varphi_1\rangle\langle\varphi_1|} \langle\varphi_2| + |\varphi_1\rangle\langle\varphi_2| |\varphi_2\rangle\langle\varphi_1| + |\varphi_2\rangle\langle\varphi_1| |\varphi_1\rangle\langle\varphi_2| + |\varphi_2\rangle\cancel{|\varphi_1\rangle\langle\varphi_2|} \langle\varphi_1| \right] = \\ &= \alpha^2 \left[|\varphi_1\rangle\langle\varphi_2| |\varphi_2\rangle\langle\varphi_1| + |\varphi_2\rangle\langle\varphi_1| |\varphi_1\rangle\langle\varphi_2| \right] = \alpha^2 \left[|\varphi_1\rangle\langle\varphi_1| + |\varphi_2\rangle\langle\varphi_2| \right] \neq \hat{H} \end{aligned}$$

$$\hat{H} = \alpha(|\varphi_1\rangle\langle\varphi_2| + |\varphi_2\rangle\langle\varphi_1|)$$

$$(1) \quad \hat{H}^\dagger = \left\{ \alpha(|\varphi_1\rangle\langle\varphi_2| + |\varphi_2\rangle\langle\varphi_1|) \right\}^\dagger = \alpha^* (|\varphi_1\rangle\langle\varphi_2|)^\dagger + \alpha^* (|\varphi_2\rangle\langle\varphi_1|)^\dagger = \alpha(|\varphi_2\rangle\langle\varphi_1|) + \alpha(|\varphi_1\rangle\langle\varphi_2|) = \hat{H}$$

$$\begin{aligned} (2) \quad \hat{H}^2 &= \hat{H}\hat{H} = \alpha(|\varphi_1\rangle\langle\varphi_2| + |\varphi_2\rangle\langle\varphi_1|)\alpha(|\varphi_1\rangle\langle\varphi_2| + |\varphi_2\rangle\langle\varphi_1|) = \\ &= \alpha^2 \left[(|\varphi_1\rangle\langle\varphi_2|)(|\varphi_1\rangle\langle\varphi_2|) + (|\varphi_1\rangle\langle\varphi_2|)(|\varphi_2\rangle\langle\varphi_1|) + (|\varphi_2\rangle\langle\varphi_1|)(|\varphi_1\rangle\langle\varphi_2|) + (|\varphi_2\rangle\langle\varphi_1|)(|\varphi_2\rangle\langle\varphi_1|) \right] = \\ &= \alpha^2 \left[|\varphi_1\rangle\langle\varphi_2| \cancel{|\varphi_1\rangle\langle\varphi_1|} \langle\varphi_2| + |\varphi_1\rangle\langle\varphi_2| |\varphi_2\rangle\langle\varphi_1| + |\varphi_2\rangle\langle\varphi_1| |\varphi_1\rangle\langle\varphi_2| + |\varphi_2\rangle\cancel{|\varphi_1\rangle\langle\varphi_2|} \langle\varphi_1| \right] = \\ &= \alpha^2 \left[|\varphi_1\rangle\langle\varphi_2| |\varphi_2\rangle\langle\varphi_1| + |\varphi_2\rangle\langle\varphi_1| |\varphi_1\rangle\langle\varphi_2| \right] = \alpha^2 \left[|\varphi_1\rangle\langle\varphi_1| + |\varphi_2\rangle\langle\varphi_2| \right] \neq \hat{H} \end{aligned}$$

$$\frac{\hat{H}^2}{\alpha^2} = |\varphi_1\rangle\langle\varphi_1| + |\varphi_2\rangle\langle\varphi_2| \stackrel{\text{def.}}{=} \hat{B}$$

$$(1) \quad \hat{B}^\dagger = \left(\frac{\hat{H}^2}{\alpha^2} \right)^\dagger = \frac{1}{(\alpha^*)^2} (\hat{H}\hat{H})^\dagger = \frac{1}{\alpha^2} \hat{H}^\dagger \hat{H}^\dagger = \frac{\hat{H}\hat{H}}{\alpha^2} = \hat{B}$$

$$\begin{aligned} (2) \quad \hat{B}^2 &= \hat{B}\hat{B} = (|\varphi_1\rangle\langle\varphi_1| + |\varphi_2\rangle\langle\varphi_2|)(|\varphi_1\rangle\langle\varphi_1| + |\varphi_2\rangle\langle\varphi_2|) = \\ &= |\varphi_1\rangle\langle\varphi_1| |\varphi_1\rangle\langle\varphi_1| + |\varphi_1\rangle\cancel{|\varphi_1\rangle\langle\varphi_2|} \langle\varphi_2| + |\varphi_2\rangle\cancel{|\varphi_2\rangle\langle\varphi_1|} \langle\varphi_1| + |\varphi_2\rangle\langle\varphi_2| |\varphi_2\rangle\langle\varphi_2| = \\ &= |\varphi_1\rangle\langle\varphi_1| + |\varphi_2\rangle\langle\varphi_2| = \hat{B} \end{aligned}$$

$$\begin{aligned} \hat{H}|\varphi_1\rangle &= \alpha(|\varphi_1\rangle\langle\varphi_2| + |\varphi_2\rangle\langle\varphi_1|)|\varphi_1\rangle = \alpha|\varphi_1\rangle\cancel{|\varphi_2\rangle\langle\varphi_1|} + \alpha|\varphi_2\rangle\langle\varphi_1|\varphi_1\rangle = \alpha|\varphi_2\rangle \text{ is not } \propto |\varphi_1\rangle \\ \hat{H}|\varphi_2\rangle &= \alpha(|\varphi_1\rangle\langle\varphi_2| + |\varphi_2\rangle\langle\varphi_1|)|\varphi_2\rangle = \alpha|\varphi_1\rangle\langle\varphi_2|\varphi_2\rangle + \alpha|\varphi_2\rangle\cancel{|\varphi_1\rangle\langle\varphi_2|} = \alpha|\varphi_1\rangle \text{ is not } \propto |\varphi_2\rangle \end{aligned}$$

$$[\hat{H}, |\varphi_1\rangle\langle\varphi_1|] = \hat{H}|\varphi_1\rangle\langle\varphi_1| - |\varphi_1\rangle\langle\varphi_1|\hat{H} = \alpha|\varphi_2\rangle\langle\varphi_1| - |\varphi_1\rangle\langle\varphi_2| \alpha^* = \alpha(|\varphi_2\rangle\langle\varphi_1| - |\varphi_1\rangle\langle\varphi_2|)$$

$$[\hat{H}, |\varphi_2\rangle\langle\varphi_2|] = \hat{H}|\varphi_2\rangle\langle\varphi_2| - |\varphi_2\rangle\langle\varphi_2|\hat{H} = \alpha|\varphi_1\rangle\langle\varphi_2| - |\varphi_2\rangle\langle\varphi_1| \alpha^* = \alpha(|\varphi_1\rangle\langle\varphi_2| - |\varphi_2\rangle\langle\varphi_1|)$$

$$\text{We see } [\hat{H}, |\varphi_2\rangle\langle\varphi_2|] = -[\hat{H}, |\varphi_1\rangle\langle\varphi_1|]$$

Trial normalized eigenket: $|\psi\rangle = \lambda_1 |\varphi_1\rangle + \lambda_2 |\varphi_2\rangle$

$$\langle\psi|\psi\rangle = |\lambda_1|^2 + |\lambda_2|^2 = 1$$

$$\hat{H}|\psi\rangle = \hat{H} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \alpha\lambda_2 \\ \alpha\lambda_1 \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = E \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \Rightarrow \begin{pmatrix} -E & \alpha \\ \alpha & -E \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$\begin{vmatrix} -E & \alpha \\ \alpha & -E \end{vmatrix} = E^2 - \alpha^2 = 0 \Rightarrow E = \pm\alpha$$

$$E = +\alpha \Rightarrow -\alpha\lambda_1 + \alpha\lambda_2 = 0 \Rightarrow \lambda_2 = \lambda_1 \Rightarrow |\psi\rangle = \begin{pmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix}$$

$$E = -\alpha \Rightarrow \alpha\lambda_1 + \alpha\lambda_2 = 0 \Rightarrow \lambda_2 = -\lambda_1 \Rightarrow |\psi\rangle = \begin{pmatrix} \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} \end{pmatrix}$$

QM B3

$$\psi(x, y, z) = \frac{1}{4\sqrt{\pi}} \frac{2z^2 - x^2 - y^2}{r^2} + \sqrt{\frac{3}{\pi}} \frac{xz}{r^2} = \frac{1}{4\sqrt{\pi}} \frac{3z^2 - r^2}{r^2} + \sqrt{\frac{3}{\pi}} \frac{xz}{r^2}$$

$$Y_{20} = \sqrt{\frac{5}{16\pi}} \frac{3z^2 - r^2}{r^2} \text{ so } \frac{3z^2 - r^2}{r^2} = \sqrt{\frac{16\pi}{5}} Y_{20}$$

$$Y_{2,\pm 1} = \mp \sqrt{\frac{15}{8\pi}} \frac{(x \pm iy)z}{r^2} \Rightarrow Y_{2,+1} = -\sqrt{\frac{15}{8\pi}} \frac{(x+iy)z}{r^2} \text{ and } Y_{2,-1} = +\sqrt{\frac{15}{8\pi}} \frac{(x-iy)z}{r^2} \Rightarrow$$

$$Y_{2,-1} - Y_{2,+1} = \sqrt{\frac{15}{8\pi}} \left(\frac{(x-iy)z}{r^2} + \frac{(x+iy)z}{r^2} \right) = \sqrt{\frac{15}{8\pi}} \left(\frac{xz - iyz}{r^2} + \frac{xz + iyz}{r^2} \right) = 2\sqrt{\frac{15}{8\pi}} \frac{xz}{r^2}$$

$$\frac{xz}{r^2} = \frac{1}{2} \sqrt{\frac{8\pi}{15}} (Y_{2,-1} - Y_{2,+1})$$

$$\begin{aligned} \psi(\mathbf{r}) &= \frac{1}{4\sqrt{\pi}} \left[\frac{3z^2 - r^2}{r^2} \right] + \sqrt{\frac{3}{\pi}} \left[\frac{xz}{r^2} \right] = \frac{1}{4\sqrt{\pi}} \left[\sqrt{\frac{16\pi}{5}} Y_{20} \right] + \sqrt{\frac{3}{\pi}} \left[\frac{1}{2} \sqrt{\frac{8\pi}{15}} (Y_{2,-1} - Y_{2,+1}) \right] = \\ &= \frac{1}{4\sqrt{\pi}} \sqrt{\frac{16\pi}{5}} [Y_{20}] + \sqrt{\frac{3}{\pi}} \frac{1}{2} \sqrt{\frac{8\pi}{15}} [Y_{2,-1} - Y_{2,+1}] = \frac{1}{\sqrt{5}} Y_{20} + \sqrt{\frac{2}{5}} [Y_{2,-1} - Y_{2,+1}] = \\ &= \frac{1}{\sqrt{5}} Y_{20} + \sqrt{\frac{2}{5}} Y_{2,-1} - \sqrt{\frac{2}{5}} Y_{2,+1} = \frac{1}{\sqrt{5}} |2,0\rangle + \sqrt{\frac{2}{5}} |2,-1\rangle - \sqrt{\frac{2}{5}} |2,+1\rangle \end{aligned}$$

$$|\psi\rangle = \frac{1}{\sqrt{5}}|2,0\rangle + \sqrt{\frac{2}{5}}|2,-1\rangle - \sqrt{\frac{2}{5}}|2,+1\rangle$$

$$\langle\psi|\psi\rangle = \left|\frac{1}{\sqrt{5}}\right|^2 + \left|\sqrt{\frac{2}{5}}\right|^2 + \left|-\sqrt{\frac{2}{5}}\right|^2 = 1$$

$$|\psi\rangle = \frac{1}{\sqrt{5}}|2,0\rangle + \sqrt{\frac{2}{5}}|2,-1\rangle - \sqrt{\frac{2}{5}}|2,+1\rangle$$

$$\langle\hat{L}^2\rangle = \langle\psi|\hat{L}^2|\psi\rangle = \left|\frac{1}{\sqrt{5}}\right|^2 \cdot 6\hbar^2 + \left|\sqrt{\frac{2}{5}}\right|^2 \cdot 6\hbar^2 + \left|-\sqrt{\frac{2}{5}}\right|^2 \cdot 6\hbar^2 = 6\hbar^2$$

$$\langle\hat{L}_z\rangle = \langle\psi|\hat{L}_z|\psi\rangle = \left|\frac{1}{\sqrt{5}}\right|^2 \cdot 0\hbar + \left|\sqrt{\frac{2}{5}}\right|^2 \cdot (-1)\hbar + \left|-\sqrt{\frac{2}{5}}\right|^2 \cdot (+1)\hbar = 0$$

$$\langle\hat{L}_+\rangle = \langle\psi|\hat{L}_+|\psi\rangle$$

$$\hat{L}_+|\psi\rangle = \frac{1}{\sqrt{5}}\hat{L}_+|2,0\rangle + \sqrt{\frac{2}{5}}\hat{L}_+|2,-1\rangle - \cancel{\sqrt{\frac{2}{5}}\hat{L}_+|2,+1\rangle} = \frac{1}{\sqrt{5}}\hbar\sqrt{6}|2,+1\rangle + \sqrt{\frac{2}{5}}\hbar\sqrt{6}|2,0\rangle$$

$$\langle\psi|\hat{L}_+|\psi\rangle = \left\{ \frac{1}{\sqrt{5}}\langle 2,0 | + \sqrt{\frac{2}{5}}\langle 2,-1 | - \sqrt{\frac{2}{5}}\langle 2,+1 | \right\} \left\{ \frac{1}{\sqrt{5}}\hbar\sqrt{6}|2,+1\rangle + \sqrt{\frac{2}{5}}\hbar\sqrt{6}|2,0\rangle \right\} =$$

$$= \frac{1}{\sqrt{5}}\sqrt{\frac{2}{5}}\hbar\sqrt{6}\langle 2,0 | 2,0 \rangle - \sqrt{\frac{2}{5}}\frac{1}{\sqrt{5}}\hbar\sqrt{6}\langle 2,+1 | 2,+1 \rangle =$$

=

$$= \frac{1}{\sqrt{5}}\sqrt{\frac{2}{5}}\hbar\sqrt{6} - \sqrt{\frac{2}{5}}\frac{1}{\sqrt{5}}\hbar\sqrt{6} = 0$$

$$|\psi\rangle = \frac{1}{4\sqrt{\pi}} \frac{3z^2 - r^2}{r^2} + \sqrt{\frac{3}{\pi}} \frac{xz}{r^2} =$$

$$= \frac{1}{4\sqrt{\pi}}(3\cos^2\theta - 1) + \sqrt{\frac{3}{\pi}} \frac{(\cancel{\sin\theta\cos\phi})(\cancel{\cos\theta})}{\cancel{\rho^2}} = \frac{1}{4\sqrt{\pi}}(3\cos^2\theta - 1) + \sqrt{\frac{3}{\pi}} \sin\theta\cos\theta\cos\phi$$

$$P = \int_{\Delta\Omega} |\psi|^2 d\Omega \approx |\psi|^2 \int_{\Delta\Omega} d\Omega = |\psi|^2 \Delta\Omega = \left[\frac{1}{4\sqrt{\pi}}(3\cos^2\theta - 1) + \sqrt{\frac{3}{\pi}} \sin\theta\cos\theta\cos\phi \right]^2 \Delta\Omega =$$

$$= \left[\frac{1}{4\sqrt{\pi}}(3\cos^2(\frac{1}{3}\pi) - 1) + \sqrt{\frac{3}{\pi}} \sin(\frac{1}{3}\pi)\cos(\frac{1}{3}\pi)\cos(\frac{1}{2}\pi) \right]^2 \cdot 10^{-3} =$$

$$= \left[\frac{1}{4\sqrt{\pi}}(3(\frac{1}{2})^2 - 1) + \sqrt{\frac{3}{\pi}} (\cancel{\frac{1}{2}\sqrt{3}})(\cancel{\frac{1}{2}})(0) \right]^2 \cdot 10^{-3} = \left[\frac{1}{4\sqrt{\pi}}(-\frac{1}{4}) \right]^2 \cdot 10^{-3} = 1.24 \times 10^{-6}$$

QM B4

$$(a) \quad \Psi(\vec{r}, 0) = \frac{1}{\sqrt{2}} [\Psi_{100}(\vec{r}) + A (\Psi_{210}(\vec{r}) + \Psi_{211}(\vec{r}) + \Psi_{21-1}(\vec{r}))]$$

$$\int |\Psi(\vec{r}, 0)|^2 d\vec{r} = \frac{1}{2} \int [|\Psi_{100}(\vec{r})|^2 + A^2 (|\Psi_{210}(\vec{r})|^2 + |\Psi_{211}(\vec{r})|^2 + |\Psi_{21-1}(\vec{r})|^2)] d\vec{r}$$

$$= \frac{1}{2} (1 + 3A^2) = 1$$

$$3A^2 = 1 \quad A = \frac{1}{\sqrt{3}}$$

(b)

$$P(r) = r^2 \int |\Psi(\vec{r}, 0)|^2 d\Omega$$

$$= \frac{r^2}{2} [R_{10}^2(r) + \frac{1}{3} R_{21}^2(r) \cdot 3] = \frac{r^2}{2} [R_{10}^2(r) + R_{21}^2(r)]$$

$$(c) \quad \langle r \rangle = \frac{1}{2} \int r^3 [R_{10}^2(r) + R_{21}^2(r)] dr$$

$$\left. \begin{aligned} & \text{since } \int |\Psi_{em}(\vec{r})|^2 d\vec{r} \\ & \int Y_{em}^* Y_{em} d\Omega = \delta_{ee} \delta_{mm} \end{aligned} \right\} = 1$$

$$= \frac{1}{2} \int r^3 \left[\frac{4}{a_0^3} e^{-2r/a_0} + \frac{r^2}{24a_0^5} e^{-r/a_0} \right] dr$$

$$= \frac{1}{2} \left[\frac{4}{a_0^3} \left(\frac{a_0}{2} \right)^4 \cdot 3! + \frac{1}{24a_0^5} \cdot \cancel{\left(\frac{a_0}{2} \right)^6} \cdot 5! \right] = \frac{a_0}{2} \left(\frac{6}{4} + 5 \right) = \frac{13}{4} a_0$$

(d) 0 and $2\hbar^2$ with probabilities $\frac{1}{2}$ and $\frac{1}{2}$

$$(e) \quad L_z = 1 : p = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$$

$$L_z = -1 : p = \frac{1}{6} \quad L_z = 0 : \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$

$$(f) \quad \Psi(\vec{r}, t) = \frac{1}{\sqrt{2}} [\Psi_{100}(\vec{r}) e^{-iE_1 t/\hbar} + \frac{1}{\sqrt{3}} (\Psi_{210} + \Psi_{211} + \Psi_{21-1}) e^{-iE_2 t/\hbar}]$$

$$\text{where } E_n = -\frac{mk^2 e^4}{2n^2 \hbar^2} \quad n=1, 2.$$

$$(g) \quad \Psi(\vec{r}, t) = \Psi_{211}(\vec{r}) e^{-iE_2 t/\hbar}$$

EM A1

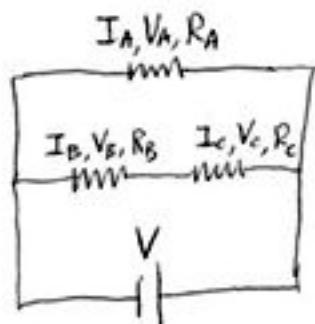
$$\Phi = EA = \frac{V}{d} A \Rightarrow \frac{d\Phi}{dt} = \frac{A}{d} \frac{dV}{dt} \Rightarrow I_{\text{displ.}} = \epsilon_0 \frac{d\Phi}{dt} = \epsilon_0 \frac{A}{d} \frac{dV}{dt} = C \frac{dV}{dt}$$
$$C = \epsilon_0 \frac{A}{d} = 8.854 \times 10^{-12} * \frac{(5 \times 10^{-2})^2}{0.5 \times 10^{-3}} = 4.425 \times 10^{-11} \text{ F} = 44.3 \text{ pF}$$
$$I_{\text{displ.}} = C \frac{dV}{dt} = 4.425 \times 10^{-11} * 500 \times 10^3 = 2.21 \times 10^{-5} \text{ A} = 22.1 \mu\text{A}$$

EM A2

$$K_i + U_i = K_f + U_f \Rightarrow K_f = U_i = U_{12} + U_{13} + U_{14} + U_{23} + U_{24} + U_{34} =$$
$$= U_{12} + U_{13} + U_{12} + U_{12} + U_{13} + U_{12} = 4U_{12} + 2U_{13} =$$
$$= 4k \frac{qq}{s} + 2k \frac{qq}{\sqrt{2}s} = k \frac{qq}{s} \left(4 + \frac{2}{\sqrt{2}} \right) = k \frac{qq}{s} (4 + \sqrt{2}) = 5.4142 k \frac{qq}{s} =$$
$$= 8.9875 \times 10^9 * 5.4142 * \frac{10^{-8} * 10^{-8}}{10^{-2}} = 8.9875 * 5.4142 * 10^{-5} = 4.866 \times 10^{-4} \text{ J}$$

$$K_{\substack{\text{four} \\ \text{particles}}} = 4.866 \times 10^{-4} \text{ J} \Rightarrow K_{\substack{\text{one} \\ \text{particle}}} = 1.217 \times 10^{-4} = \frac{1}{2} mv^2 = \frac{1}{2} (10^{-3}) v^2 \Rightarrow$$
$$2.433 \times 10^{-1} = v^2 \Rightarrow v = \sqrt{2.433 \times 10^{-1}} = 0.493 \text{ m/s}$$

EM A3



$$I = \frac{V}{R}, \quad R = R_1 + R_2, \quad P = IV$$

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

(a) $R_{BC} = R_B + R_C = 20 + 10 = 30 \Omega$

$$\Rightarrow I_A = \frac{10}{20} = \frac{1}{2} A$$

$$I_B = I_C = \frac{10}{30} = \frac{1}{3} A$$

(b) $V_A = 10 V$

$$V_B = \frac{1}{3} \cdot 20 = \frac{20}{3} V$$

$$V_C = \frac{1}{3} \cdot 10 = \frac{10}{3} V$$

(c) $P_A = 10 \cdot \frac{1}{2} = \frac{10}{2} W$ Lightbulb A

$$P_B = \frac{20}{3} \cdot \frac{1}{3} = \frac{20}{9} W \Rightarrow \text{has the highest}$$

$$P_C = \frac{10}{3} \cdot \frac{1}{3} = \frac{10}{9} W \quad \text{brightness}$$

(d) $I = I_A + I_B = \frac{1}{2} + \frac{1}{3} = \frac{5}{6} A$

or. $\frac{1}{R_{ABC}} = \frac{1}{20} + \frac{1}{30} = \frac{3+2}{60} = \frac{5}{60} = \frac{1}{12}$

$$\Rightarrow R_{ABC} = 12 \Omega$$

$$I = \frac{10}{12} = \frac{5}{6} A$$

EM A4

Consider two concentric spherical metal shells of radii r_1 and r_2 ($r_2 > r_1$). If the outer shell has a charge q and the inner shell is grounded, what is its charge?

Let q' be the charge on the inner shell. The potential at the inner shell is the sum of the potentials due to q and q' , respectively:

$$V = \frac{q}{r_2} + \frac{q'}{r_1}$$

which is zero since the inner shell is grounded. Therefore

$$q' = -\frac{r_1}{r_2} q.$$

EM B1

Apply $L = \frac{N\Phi_B}{i}$ to calculate L .

In the air the magnetic field is $B_{\text{Air}} = \frac{\mu_0 Ni}{W}$. In the liquid, $B_L = \frac{\mu Ni}{W}$.

$$\Phi_B = BA = B_L A_L + B_{\text{Air}} A_{\text{Air}} = \frac{\mu_0 Ni}{W} [(D - d)W] + \frac{K\mu_0 Ni}{W} (dW) = \mu_0 Ni [(D - d) + Kd].$$

$$L = \frac{N\Phi_B}{i} = \mu_0 N^2 [(D - d) + Kd] = L_0 - L_0 \frac{d}{D} + L_f \frac{d}{D} = L_0 + \left(\frac{L_f - L_0}{D} \right) d.$$

$$d = \left(\frac{L - L_0}{L_f - L_0} \right) D, \text{ where } L_0 = \mu_0 N^2 D, \text{ and } L_f = K\mu_0 N^2 D.$$

$$\text{Using } \epsilon_r = \chi_m + 1 \text{ we can find the inductance for any height } L = L_0 \left(1 + \chi_m \frac{d}{D} \right).$$

Height of Fluid	Inductance
$d = D/4$	$L / L_0 = 1.0016$
$d = D/2$	$L / L_0 = 1.0032$
$d = 3D/4$	$L / L_0 = 1.0048$
$d = D$	$L / L_0 = 1.0064$

EM B2

Analysis

We have chosen two parts in the circuit in the picture from assigned task. We „know“ the amplitude of the voltage of each of them so we can solve each part as a separate circuit. We use Ohm’s law for an alternating circuit. The current is the same in both of the circuits, so we can formulate the capacity from the ratio of voltages.

We gain the amplitude of current from Ohm’s law which we use for the whole circuit.

Expressing of a voltage U_{m1} and U_{m2}

A coil with an inductance L and a resistor with resistance R are connected to the part of the circuit which belongs to voltage U_{m1} . The coil and the resistor are connected in series.

We formulate impedance Z_1 of this part of the circuit:

$$Z_1 = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + X_L^2}.$$

And substitute it to Ohm’s law:

$$U_{m1} = I_m Z_1 = I_m \sqrt{R^2 + X_L^2} = I_m \sqrt{R^2 + (\omega L)^2}.$$

Formulas for the second part of the circuit, to which belongs voltage U_{m2} are similar. In this circuit we have a resistor with resistivity R and a capacitor with an unknown capacity C :

$$Z_2 = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + X_C^2}$$

$$U_{m2} = I_m Z_2 = I_m \sqrt{R^2 + X_C^2} = I_m \sqrt{R^2 + \frac{1}{(\omega C)^2}}.$$

Expressing a capacity C and an amplitude of a current I_m

From the assignment we know the ratio of both voltages. So we replace it with formulas which we derive from voltages in previous part:

$$\frac{U_1}{U_2} = \frac{I_m \sqrt{R^2 + (\omega L)^2}}{I_m \sqrt{R^2 + \frac{1}{(\omega C)^2}}}$$

The amplitude of current I_m is the same for both formulas so we can reduce it.

$$\frac{1}{2} = \frac{\sqrt{R^2 + (\omega L)^2}}{\sqrt{R^2 + \frac{1}{(\omega C)^2}}}$$

Express unknown capacity C :

$$\frac{1}{4} = \frac{R^2 + (\omega L)^2}{R^2 + \left(\frac{1}{\omega C}\right)^2}$$

$$R^2 + \left(\frac{1}{\omega C}\right)^2 = 4(R^2 + (\omega L)^2)$$

$$\left(\frac{1}{\omega C}\right)^2 = 3R^2 + 4(\omega L)^2$$

$$(\omega C)^2 = \frac{1}{3R^2 + 4(\omega L)^2}$$

$$C = \frac{1}{\omega \sqrt{3R^2 + 4(\omega L)^2}}.$$

We should express the amplitude of the current from Ohm's law for the whole circuit. From the assignment we know the amplitude of the voltage of the power supply U_m . There are all three segments in the circuit so we need all three quantities for the formula of impedance – resistance, capacitive reactance and inductive reactance.

$$I_m = \frac{U_m}{Z} = \frac{U_m}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}$$

Numerical solution

Capacity of the capacitor:

$$C = \frac{1}{\omega \sqrt{3R^2 + 4(\omega L)^2}} = \frac{1}{2 \cdot \pi \cdot 50 \sqrt{3 \cdot 50^2 + 4(2 \cdot \pi \cdot 50 \cdot 0.1)^2}} \text{ F}$$

$$C \doteq 30 \cdot 10^{-6} \text{ F} = 30 \mu\text{F}$$

Amplitude of the current:

$$I_m = \frac{U_m}{Z} = \frac{U_m}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}$$

$$I_m = \frac{300}{\sqrt{50^2 + (2 \cdot \pi \cdot 50 \cdot 0.1 - \frac{1}{2 \cdot \pi \cdot 50 \cdot 30 \cdot 10^{-6}})^2}} \text{ A}$$

$$I_m \doteq 3.3 \text{ A}$$

EM - B3

- (a) Apply the integrated version of Faraday's law to the closed loop \mathcal{L} made up of the segment of the circle connecting the moving end of the rod and the point C, moving counter clockwise (when viewed from above) from that end to C, the segment from C to A containing the resistor R and then the rod itself:

$$\oint_{\mathcal{L}} d\vec{r} \cdot \vec{E} = -\frac{d}{dt} \Phi_B$$

where $\Phi_B(t)$ is the flux of magnetic field passing through that loop. Evaluating the left and right hand sides of this equation:

$$IR = \frac{\omega r^2}{2} B \quad \text{or} \quad I = \frac{r^2 B \omega}{2R}$$

- (b) As found in (a), I is positive.
- (c) Consider a segment of the rod of length dx a distance x from the pivot. Because of the flowing current this segment will contain a charge dQ moving at velocity v where $I = dQ/(dx/v) = vdQ/dx$. The torque τ exerted by the magnetic field on the rod carrying the current I is then given by:

$$\tau = \int_0^r xv B \frac{dQ}{dx} dx = IB \int_0^r x dx = \frac{IBr^2}{2} = \frac{r^4 B^2 \omega}{4R}$$

pointed downward. Thus, $-\vec{\tau}$ must be applied to the rod to maintain its rotational motion.

EM B4

The weight \vec{F}_G and the tensile force from the thread \vec{F}_t act on each ball in the air.

The balls are charged thus they repeal each other by the electric force \vec{F}_e .

For the magnitude of these forces it holds:

$$F_G = mg$$

$$F_e = \frac{1}{4\pi\epsilon_0} \frac{Q^2}{r^2},$$

where r is the distance between the balls.

The balls are stationary thus the net force of these three forces must be zero.

We can express the tangent function of the angle α from the “purple“ right-angled triangle:

$$\begin{aligned} \operatorname{tg}\alpha &= \frac{F_e}{F_G} = \frac{\frac{1}{4\pi\epsilon_0} \frac{Q^2}{r^2}}{mg} \\ \operatorname{tg}\alpha &= \frac{1}{4\pi\epsilon_0} \frac{Q^2}{mgr^2} \end{aligned} \quad (*)$$

If we immerse the balls into benzene a new force, the buoyant force \vec{F}_{VZ} , appears.

The weight stays the same but the repulsive electric force \vec{F}_e will be ϵ_r -times smaller. The tensile force will be changed as well.

$$F_G = mg$$

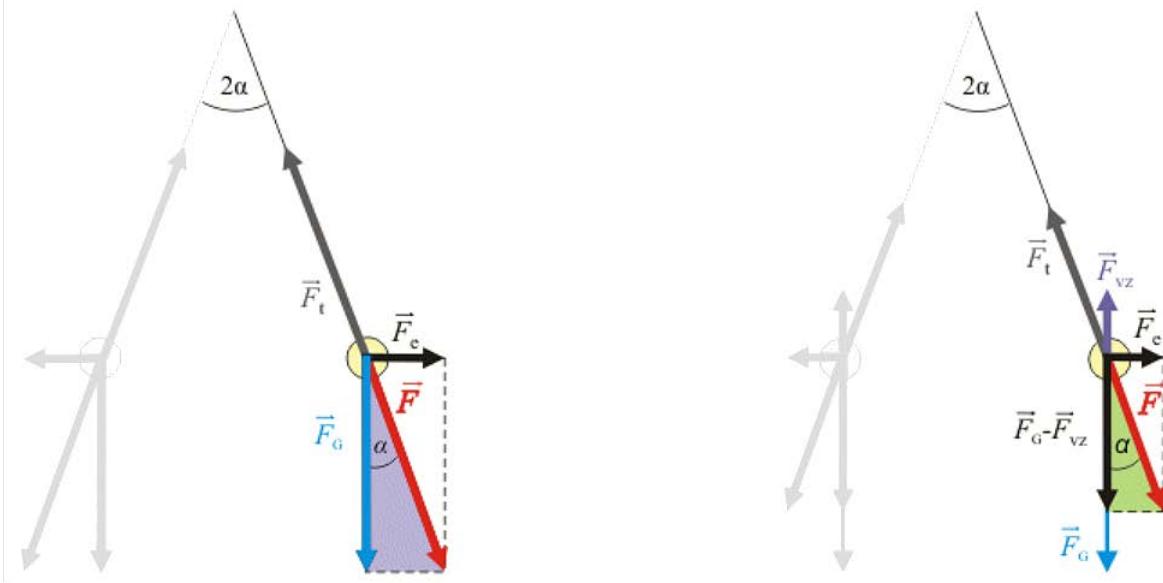
$$F_e = \frac{1}{4\pi\epsilon_0\epsilon_r} \frac{Q^2}{r^2}$$

$$F_{VZ} = V\varrho_b g$$

The net force of these four forces must remain zero.

This time we express the $\operatorname{tg} \alpha$ from the “green“ right-angled triangle:

$$\operatorname{tg}\alpha = \frac{F_e}{F_G - F_{VZ}} = \frac{\frac{1}{4\pi\epsilon_0\epsilon_r} \frac{Q^2}{r^2}}{mg - V\varrho_b g}$$



Now we compare the expressions (*) and (**) for the tangent function of the angle α :

$$\frac{1}{4\pi\epsilon_0} \frac{Q^2}{mgr^2} = \frac{1}{4\pi\epsilon_0\epsilon_r} \frac{Q^2}{r^2g(m - V\varrho_b)}$$

We multiply both sides of the equation by

$$4\pi\epsilon_0 gr^2$$

and divide them by Q^2

$$\frac{1}{m} = \frac{1}{\epsilon_r} \frac{1}{m - V\varrho_b}$$

For the mass of one ball m we insert $m = V\rho$ and obtain:

$$\frac{1}{V\varrho} = \frac{1}{\epsilon_r} \frac{1}{V\varrho - V\varrho_b}$$

From this equation we express the unknown density of the ball ρ as

$$\frac{1}{\varrho} = \frac{1}{\epsilon_r} \frac{1}{\varrho - \varrho_b}$$

$$\varrho = \epsilon_r (\varrho - \varrho_b)$$

We multiply the expression in the brackets by ϵ_r and move the terms containing the unknown ρ to the left-hand side.

$$\epsilon_r \varrho - \varrho = \epsilon_r \varrho_b$$

$$\varrho (\epsilon_r - 1) = \epsilon_r \varrho_b$$

$$\varrho = \frac{\varrho_b \epsilon_r}{\epsilon_r - 1}$$

Numerical values

$\rho_b = 879 \text{ kg m}^{-3}$the density of benzene

$\varepsilon_r = 2.3$

$\rho = ? \text{ (kg m}^{-3}\text{)}$

$$\rho = \frac{\rho_b \varepsilon_r}{\varepsilon_r - 1} = \frac{2.3 \cdot 879}{2.3 - 1} \text{ kg m}^{-3} = 1600 \text{ kg m}^{-3}$$