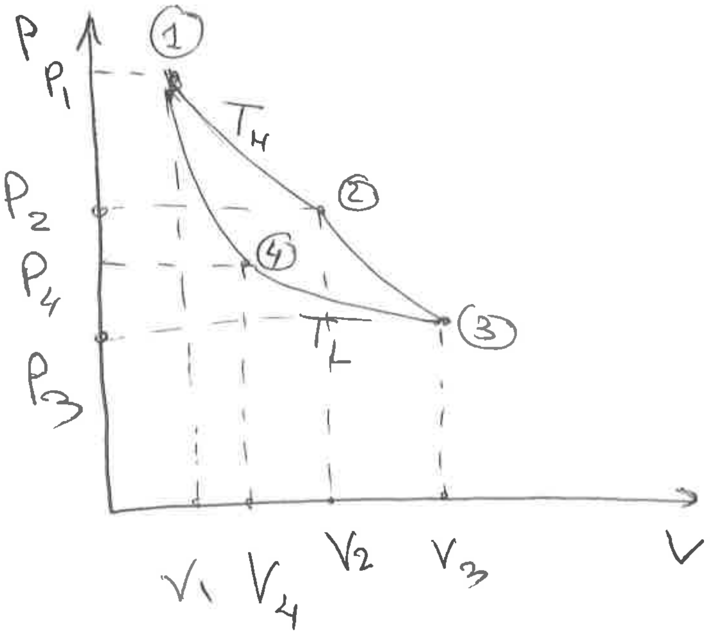
TH A1

Easy: Carnot cycle

Derive the expression for the efficiency, defined as the total work done over the total heat supplied, for a Carnot cycle which uses a monoatomic ideal gas as an operating substance. Use the equation of state for the gas $PV = nRT$ and the internal energy $U = \frac{3}{2}nRT$.



Consider isothermal segments $1 \to 2$ and $3 \to 4$. Since $T = const$, $dT = 0$ and $dU = \frac{3}{2} nR\,dT = 0$.

Thus, for $1 \to 2$,

$$dQ = dW = nRT_H \frac{dV}{V}$$

after integration:

$$\Delta Q_{12} = nRT_H \int_{V_1}^{V_2} \frac{dV}{V} = nRT_H \ln \frac{V_2}{V_1}$$

Similarly

$$\Delta Q_{34} = nRT_L \ln \frac{V_4}{V_3}$$

Consider adiabatic segments $2 \to 3$ and $4 \to 1$.

Here, $dQ = 0 = dU + PdV = \frac{3}{2} nR\,dT + PdV$.

$\frac{3}{2} nR\,dT = -PdV$, divide this by the eq. of state $nRT = PV$:

$$\frac{3}{2}\frac{dT}{T} = -\frac{dV}{V} \Rightarrow VT^{3/2} = const.$$

Therefore, for $2 \to 3$ and $4 \to 1$ we have

$$T_L V_3^{2/3} = T_H V_2^{2/3} \text{ and } T_L V_4^{2/3} = T_H V_1^{2/3}$$

from where it follows that $\frac{V_3}{V_4} = \frac{V_2}{V_1}$.

For entire cycle: $\Delta U_{tot} = \Delta Q_{tot} - \Delta W_{tot} = 0$.

$$\Delta W_{tot} = \Delta Q_{tot} = \Delta Q_{12} + \Delta Q_{34}.$$

$$\eta = \frac{\Delta W_{tot}}{\Delta Q_{12}} = 1 + \frac{\Delta Q_{34}}{\Delta Q_{12}} = 1 - \frac{T_L}{T_H}\frac{\ln(V_3/V_4)}{\ln(V_2/V_1)} = 1 - \frac{T_L}{T_H}$$

received.

TH A2

Easy: Enthalpy and heat capacity

Prove that the C_p for an ideal gas is independent of pressure. Reminder: heat capacity at constant pressure can be defined as $C_p = (\partial H/\partial T)_P$.

Enthalpy is defined as $H = U + PV$

Compute C_p:

$$C_p = \left(\frac{\partial H}{\partial T}\right)_P = \left(\frac{\partial (U+PV)}{\partial T}\right)_P = \left(\frac{\partial U}{\partial T}\right)_P + P\left(\frac{\partial V}{\partial T}\right)_P$$

For an ideal gas $PV = nRT$, so $P\left(\frac{\partial V}{\partial T}\right)_P = nR$, and R is not a function of pressure.

The term $\left(\frac{\partial U}{\partial T}\right)_P$ also does not depend on pressure because U is only a function of temperature.

Thus, C_p does not depend on pressure.

SOLUTIONS

Thermodynamics problems

TH A3

Easy: First Law

The internal energy for 1 kg for a certain gas is given by $U = 0.17\,T + C$ where T is the gas temperature in Kelvin, and C is a constant. The gas is heated in a rigid container (i.e. at constant volume) from a temperature of 40 °C to 316 °C. Compute the amount of work and heat flow into the system.

According to the First Law,

$$\Delta U = W + Q$$

Since the container is rigid, $\Delta V = 0$ and, therefore, $\underline{W = 0}$.

Thus

$$Q = \Delta U = U(T_1) - U(T_0)$$

$$= (0.17\,T_1 \cancel{+C}) - (0.17\,T_0 \cancel{+C}) = 0.17\,(T_1 - T_0)$$

$$\underline{\approx 46{,}75\ \mathrm{J}}$$

TH A4

A4 A large number of non-interacting particles is in equilibrium with a thermal bath of temperature 300 K. The particles have only three energy levels: $E_1 = 20\text{ meV}$, $E_2 = 30\text{ meV}$, and $E_3 = 40\text{ meV}$. Calculate the average energy of a particle.

$$Z = \Sigma_i \exp(-\beta E_i) = \Sigma_i \exp(-E_i / k_\mathrm{B} T)$$
$$k_\mathrm{B} T = 1.381 \times 10^{-23} * 300 / 1.602 \times 10^{-19} = 25.8614\text{ meV}$$
$$Z = \exp(-E_1 / k_\mathrm{B} T) + \exp(-E_2 / k_\mathrm{B} T) + \exp(-E_3 / k_\mathrm{B} T) =$$
$$= \exp(-10 / 25.8614) + \exp(-20 / 25.8614) + \exp(-50 / 25.8614) =$$
$$= 0.679311 + 0.461463 + 0.144658 = 1.28543$$
$$p_1 = \exp(-E_1 / k_\mathrm{B} T) / Z = 0.528469$$
$$p_2 = \exp(-E_2 / k_\mathrm{B} T) / Z = 0.358995$$
$$p_3 = \exp(-E_3 / k_\mathrm{B} T) / Z = 0.112537$$
$$\langle E \rangle = p_1 E_1 + p_2 E_2 + p_3 E_3 = 18.0914\text{ meV}$$

TH B1

Difficult: Thermodynamic potentials

Consider mixing 100 g of water at 300 K with 50 g of water at 400 K. Calculate the final equilibrium temperature if the specific heat c of water per gram is 1 cal/g/K. Calculate the change in entropy for this irreversible process.

The system is assumed to be in a thermally insulated vessel. From the first law : $dU + PdV = 0$.

The PdV term: we can place the system into a rigid vessel and enforce $dV = 0$. But we can also recall that for fluids dV can be neglected to a good approximation. In both cases, PdV term is dropped. The heat capacity c can be thought as c_V.

Now $dU = mc\,dT$ for each of the two fluids.

Therefore

$$\Delta U = 0 = m_1 c (T_f - T_1) + m_2 c (T_f - T_2)$$

where $m_1 = 0.1$ kg, $m_2 = 0.05$ kg, $T_1 = 300$ K, $T_2 = 400$ K and T_f is the final temperature of the mixture.

From the above we find $T_f = \dfrac{m_1 T_1 + m_2 T_2}{m_1 + m_2} = 333$ K.

Next, consider heating the m_1 water from T_1 to T_f. When temperature changes by dT, entropy gain is

$dS_1 = m_1 c \dfrac{dT}{T}$. For $T_1 \to T_f$, $\Delta S_1 = m_1 c \ln \dfrac{T_f}{T_1}$ after integration.

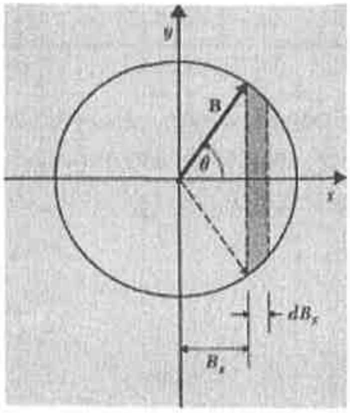
Similarly, $dS_2 = \dfrac{m_2 c\, dT}{T} \Rightarrow \Delta S_2 = m_2 c \ln \dfrac{T_f}{T_2}$.

Finally, $\Delta S_{tot} = \Delta S_1 + \Delta S_2 = 10.54 - 9.12 \simeq 1.42 \ \dfrac{cal}{deg}$.

TH B2

Difficult: Probability

A two-dimensional vector $\boldsymbol{B}$ of constant length $B = |\boldsymbol{B}|$ is equally likely to point in any direction specified by the angle θ. What is the probability that the x-component of this vector lies between B_x and $B_x + dB_x$?



The x-component of the vector is given by $B_x = B\cos\theta$.

The relation between the range dB_x and the corresponding range ~~of~~ $d\theta$ is found as

$$\left|\frac{dB_x}{d\theta}\right| = \left|-B\sin\theta\right| \quad \text{i.e.} \quad d\theta = \frac{1}{B\sin\theta}\, dB_x$$

The probability for the vector to point in the direction between θ and $\theta + d\theta$ is $\frac{d\theta}{2\pi}$.

The probability that the B_x is between B_x and $B_x + dB_x$ is equal to the probability that the vector is pointing in the direction of corresponding $\theta \to \theta + d\theta$, or symmetric direction with respecto to x-axis.

Thus:

$$P(B_x)\,dB_x = 2 \cdot \frac{1}{2\pi} \underbrace{\frac{dB_x}{|B\sin\theta|}}_{d\theta} = \frac{dB_x}{\pi B|\sin\theta|}.$$

Probability density — two-fold symmetry

compute $|\sin\theta| = \sqrt{1-\cos^2\theta} = \left[1 - \left(\frac{B_x}{B}\right)^2\right]^{1/2}$

Finally:

$$P(B_x)\,dB_x = \begin{cases} \dfrac{dB_x}{\pi\sqrt{B^2 - B_x^2}} & \text{for } B_x \in [-B, B] \\ 0 & \text{otherwise.} \end{cases}$$

TH B3

Difficult: Work

Show that the work done by a gas under arbitrary changes of temperature and pressure can be determined in terms of the coefficient of volume expansion at constant pressure α_P and the isothermal compressibility coefficient κ_T. As a corollary show that for an isochoric (constant volume) process

$$\left(\frac{\partial P}{\partial T}\right)_V = \frac{\kappa_T}{\alpha}$$

Verify this for an ideal gas. Reminder: the involved coefficients are defined as $\alpha_P = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P$ and $\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T$.

The work done is given by PdV.
Volume can be taken as a function of T and P.
Then, under arbitrary changes of T and P,

$$dW = P\,dV(T,P) = P\Big(\underbrace{\left(\frac{\partial V}{\partial T}\right)_P}_{= V\alpha_P} dT + \underbrace{\left(\frac{\partial V}{\partial P}\right)_T}_{= -\kappa_T V} dP\Big)$$

$$= P\left(V\alpha_P\,dT - V\kappa_T\,dP\right) = PV\left(\alpha_P\,dT - \kappa_T\,dP\right). \quad (*)$$

Thus, knowing the coefficients α_P and κ_T, one can always determine the work done under arbitrary changes dT and dP.

For an isochoric process $dV=0$, and $dW=0$.
From $(*)$ at $V = \text{const}$ we immediately get

$$\left(\frac{\partial P}{\partial T}\right)_V = \frac{\alpha_P}{\kappa_T}$$

For an ideal gas:

$$\left(\frac{\partial P}{\partial T}\right)_V = \left(\frac{\partial\left(\frac{nRT}{V}\right)}{\partial T}\right)_V = \frac{nR}{V}$$

These are equal for an ideal gas.

$$\alpha_P = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P = \frac{1}{V}\left(\frac{\partial \frac{nRT}{P}}{\partial T}\right)_P = \frac{nR}{PV} = \frac{1}{T}$$

$$\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T = -\frac{1}{V}\left(\frac{\partial \frac{nRT}{P}}{\partial P}\right)_T = +\frac{1}{V}\frac{nRT}{P^2} = \frac{1}{P}$$

$$\rightarrow \frac{\alpha_P}{\kappa_T} = \frac{1/T}{1/P} = P/T$$

QM A1

QM-eary 2 tyqwd mwe;'

$$S_+ = \frac{\hbar}{2}(\sigma_x + i\sigma_y) = \frac{\hbar}{2}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right] = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$S_- = S_+^\dagger = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$S_+\alpha = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

$$S_-\alpha = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hbar\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hbar\beta$$

$$S_+\beta = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hbar\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hbar\alpha$$

$$S_-\beta = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$S_+S_-\alpha = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\hbar\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hbar^2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hbar^2\alpha$$

Aug 2018

QM A2

Answers

QM easy 1.

(a) yes (H commutes with P, energy eigenstates are nondegenerate).

parity eigenvalues: ± 1

(b) $x \to -\infty$: $\psi(x) = A e^{ikx} + B e^{-ikx}$

$$k = \frac{\sqrt{2mE}}{\hbar}$$

$k|A|^2$: incident flux

$k|B|^2$: reflected flux

$x \to \infty$ $\psi(x) = C e^{ikx}$

$k|C|^2$: transmitted flux

$\psi(x)$ is <u>not</u> a parity eigenstates because of degeneracy; $\psi(x)$ is parity-mixed

QM A3

A3 A wavefunction in one dimension is given by

$$\psi(x) = \begin{cases} -C & \text{for } -a < x < 3a \\ 0 & \text{elsewhere} \end{cases}$$

where C and a are positive constants. Calculate the expectation value of the parity operator.

__Answer:__

$$\psi(x) = \begin{cases} -C & \text{for } -a < x < 3a \\ 0 & \text{elsewhere} \end{cases}$$

Normalize

$$\int_{-a}^{3a} (-C)^2\, dx = 1 \quad \Rightarrow \quad 4aC^2 = 1 \quad \Rightarrow \quad C = \frac{1}{\sqrt{4a}}$$

$$\langle \hat{P} \rangle = \int_{-\infty}^{\infty} \psi(x) \hat{P} \psi(x) dx = \int_{-\infty}^{\infty} \underbrace{-C}_{-a \text{ to } 3a}\ \underbrace{-C}_{-3a \text{ to } a}\, dx = C^2 \int_{-a}^{a} dx = \frac{1}{4a} 2a = \frac{1}{2}$$

QM A4

A4 The spherical harmonics are orthonormal; we have

$$\oiint Y^*_{\ell,m}(\theta,\phi) Y_{\ell',m'}(\theta,\phi)\, d\Omega = \delta_{\ell\ell'} \delta_{mm'}$$

where $d\Omega$ is an infinitesimal amount of solid angle, and the integral is taken over all solid angle. Use this expression to demonstrate that $Y_{1,0}$ and $Y_{1,1}$ are orthogonal.

$$\oiint Y_{1,0}(\theta,\phi) Y_{1,1}(\theta,\phi)\, d\Omega = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \left(\sqrt{\frac{3}{4\pi}} \cos\theta \right) \left(-\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\phi} \right) \sin\theta\, d\theta\, d\phi =$$

$$= -\left(\sqrt{\frac{3}{4\pi}} \right) \left(\sqrt{\frac{3}{8\pi}} \right) \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} (\cos\theta) \left(\sin\theta\, e^{i\phi} \right) \sin\theta\, d\theta\, d\phi = K \int_{\theta=0}^{\pi} \sin^2\theta \cos\theta\, d\theta\ \underbrace{\int_{\phi=0}^{2\pi} e^{i\phi} d\phi}_{=0} = 0$$

QM B1

QM hwrd 1

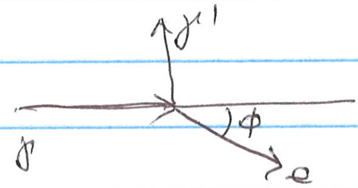
(a) $\lambda' - \lambda = \lambda_c (1 - \cos\theta)$

for $\theta = 90°$ $\quad \lambda' = \lambda + \lambda_c = \dfrac{1240\,\text{keV}\cdot\text{pm}}{200\,\text{keV}} + 2.43\,\text{pm}$
$= 8.63\,\text{pm}$

$$E' = \frac{1240\,\text{keV}\cdot\text{pm}}{8.63\,\text{pm}} = 143.7\,\text{keV}$$

(b) $K_e = E - E' = 200 - 143.7 = 56.3\,\text{keV}$

(c)



$p_{\gamma'} = p_e \cos\phi$

$\dfrac{E}{c} = p_e \cos\phi$

$$p_e = \left(\frac{E_e^2}{c^2} - m^2c^2\right)^{1/2} = (567.3^2 - 511^2)^{1/2} = 246.4\,\frac{\text{keV}}{c}$$

$$\cos\phi = \frac{E}{p_e c} = \frac{200\,\text{keV}}{246.4\,\text{keV}} \rightarrow \phi = 35.7°$$

angle between $\vec{p}_{\gamma'}$ and $\vec{p}_e$ is $125.7°$

(d) nonrel. treatment gives

$p_e = \sqrt{2mK_e} = 239.9\,\text{keV}/c$ and $\phi = 33.5°$

angle $= 123.5°$

(1.8% accuracy).

QM B2

B2 *NOTE: In this problem, we encounter infinitely large matrices, We will write these by only specifying the 4 by 4 block in the upper left corner, as in* $\begin{pmatrix} ? & ? & ? & ? & \cdots \\ ? & ? & ? & ? & \\ ? & ? & ? & ? & \\ ? & ? & ? & ? & \\ \vdots & & & & \ddots \end{pmatrix}$. *For instance, the identity operator is written as* $\hat{I} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \\ \vdots & & & & \ddots \end{pmatrix}$.

The stationary states of the harmonic oscillator are defined by $\hat{H}\,|\,n\rangle = (n+\tfrac{1}{2})\hbar\omega\,|\,n\rangle$.

The annihilation operator $\hat{a}$ of the harmonic oscillator is defined by $\hat{a} = \frac{\beta}{\sqrt{2}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)$ (with $\beta^2 = m\omega/\hbar$). The operation of the annihilation operator is $\hat{a}\,|\,n\rangle = \sqrt{n}\,|\,n-1\rangle$. Thus, in the $|\,n\rangle$ basis, the annihilation operator's matrix is $\hat{a} = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & \\ 0 & 0 & 0 & \sqrt{3} & \\ 0 & 0 & 0 & 0 & \\ \vdots & & & & \ddots \end{pmatrix}$

a. Explain why $\hat{a}^\dagger = \hat{a}^{\mathrm{T}}$, where T means matrix transposition.
b. Find the matrix for $\hat{a}^\dagger$.
c. Find the matrix for $\hat{x}$.
d. Find the matrix for $\hat{p}$.
e. Find the matrix for $\hat{x}\hat{p}$.
f. Explain why $\hat{p}\hat{x} = [(\hat{x}\hat{p})^{\mathrm{T}}]^*$, where T means matrix transposition.
g. Find the matrix for $\hat{p}\hat{x}$.
h. Find the matrix for $[\hat{x},\hat{p}]$ and comment on your answer.

Part a.
In matrix algebra, taking the Hermitian conjugate equals transposition of the matrix followed by taking its complex conjugate (or the other way around).
Because $\hat{a}$ is real-valued, transposition alone gives its Hermitian conjugate: $\hat{a}^\dagger = \hat{a}^{\mathrm{T}}$

Part b.

$$\hat{a}^{\dagger} = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & \\ 0 & 0 & 0 & \sqrt{3} & \\ 0 & 0 & 0 & 0 & \\ \vdots & & & & \ddots \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & 0 & 0 & \\ 0 & \sqrt{2} & 0 & 0 & \\ 0 & 0 & \sqrt{3} & 0 & \\ \vdots & & & & \ddots \end{pmatrix}$$

Part c. and Part d.

We have

$\hat{a} = \frac{\beta}{\sqrt{2}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)$ and $\hat{a}^{\dagger} = \frac{\beta}{\sqrt{2}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)$ so

so

$$\hat{x} = \frac{1}{\beta\sqrt{2}}\left(\hat{a} + \hat{a}^{\dagger}\right)$$

$$\hat{p} = \frac{m\omega}{i}\frac{1}{\beta\sqrt{2}}\left(\hat{a} - \hat{a}^{\dagger}\right)$$

and so

$$\hat{x} = \frac{1}{\beta\sqrt{2}} \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ \sqrt{1} & 0 & \sqrt{2} & 0 & \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \\ 0 & 0 & \sqrt{3} & 0 & \\ \vdots & & & & \ddots \end{pmatrix} ; \qquad \hat{p} = \frac{m\omega}{i}\frac{1}{\beta\sqrt{2}} \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ -\sqrt{1} & 0 & \sqrt{2} & 0 & \\ 0 & -\sqrt{2} & 0 & \sqrt{3} & \\ 0 & 0 & \sqrt{3} & 0 & \\ \vdots & & & & \ddots \end{pmatrix}$$

Part e.

$$\hat{x}\hat{p} = \frac{1}{\beta\sqrt{2}}\frac{m\omega}{i}\frac{1}{\beta\sqrt{2}} \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ \sqrt{1} & 0 & \sqrt{2} & 0 & \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \\ 0 & 0 & \sqrt{3} & 0 & \\ \vdots & & & & \ddots \end{pmatrix} \circ \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ -\sqrt{1} & 0 & \sqrt{2} & 0 & \\ 0 & -\sqrt{2} & 0 & \sqrt{3} & \\ 0 & 0 & -\sqrt{3} & 0 & \\ \vdots & & & & \ddots \end{pmatrix} = \frac{1}{i}\frac{m\omega}{2\beta^2} \begin{pmatrix} -1 & 0 & \sqrt{2} & 0 & \cdots \\ 0 & -1 & 0 & \sqrt{6} & \\ -\sqrt{2} & 0 & -1 & 0 & \\ 0 & -\sqrt{6} & 0 & -1 & \\ \vdots & & & & \ddots \end{pmatrix}$$

Part f.

The definition of Hermitian conjugate is $\hat{M}^{\dagger} = (\hat{M}^{\mathrm{T}})^{*}$, so $\hat{p}\hat{x} = \hat{p}^{\dagger}\hat{x}^{\dagger} = (\hat{x}\hat{p})^{\dagger} = [(\hat{x}\hat{p})^{\mathrm{T}}]^{*}$

Thus, to find *px*, we may transpose *xp* which we calculated in the previous part, and then take its complex conjugate.

Part g.

$$\hat{p}\hat{x} = ((\hat{x}\hat{p})^{\mathrm{T}})^* = \left(\frac{1}{i}\frac{m\omega}{2\beta^2}\begin{pmatrix} -1 & 0 & -\sqrt{2} & 0 & \cdots \\ 0 & -1 & 0 & -\sqrt{6} & \\ \sqrt{2} & 0 & -1 & 0 & \\ 0 & \sqrt{6} & 0 & -1 & \\ \vdots & & & & \ddots \end{pmatrix} \right)^* = \frac{1}{-i}\frac{m\omega}{2\beta^2}\begin{pmatrix} -1 & 0 & -\sqrt{2} & 0 & \cdots \\ 0 & -1 & 0 & -\sqrt{6} & \\ \sqrt{2} & 0 & -1 & 0 & \\ 0 & \sqrt{6} & 0 & -1 & \\ \vdots & & & & \ddots \end{pmatrix}$$

Part h.

$$\hat{x}\hat{p} - \hat{p}\hat{x} = \left\{\frac{1}{i}\frac{m\omega}{2\beta^2}\right\}\begin{pmatrix} -1 & 0 & \sqrt{2} & 0 & \cdots \\ 0 & -1 & 0 & \sqrt{6} & \\ -\sqrt{2} & 0 & -1 & 0 & \\ 0 & -\sqrt{6} & 0 & -1 & \\ \vdots & & & & \ddots \end{pmatrix} - \left\{\frac{1}{-i}\frac{m\omega}{2\beta^2}\right\}\begin{pmatrix} -1 & 0 & -\sqrt{2} & 0 & \cdots \\ 0 & -1 & 0 & -\sqrt{6} & \\ \sqrt{2} & 0 & -1 & 0 & \\ 0 & \sqrt{6} & 0 & -1 & \\ \vdots & & & & \ddots \end{pmatrix} = \left\{\frac{1}{i}\frac{m\omega}{2\beta^2}\right\}\begin{pmatrix} -2 & 0 & 0 & 0 & \cdots \\ 0 & -2 & 0 & 0 & \\ 0 & 0 & -2 & 0 & \\ 0 & 0 & 0 & -2 & \\ \vdots & & & & \ddots \end{pmatrix}$$

Since $\beta^2 = m\omega/\hbar$, we have $\frac{1}{i}\frac{m\omega}{2\beta^2} = \frac{1}{i}\frac{m\omega}{2}\frac{\hbar}{m\omega} = \frac{1}{i}\frac{\hbar}{2} = -\tfrac{1}{2}i\hbar$, and $\hat{x}\hat{p} - \hat{p}\hat{x} = -\tfrac{1}{2}i\hbar(-2\hat{I}) = i\hbar\hat{I}$

So we find $[\hat{x},\hat{p}] = i\hbar\hat{I}$, which is a general property about the position and momentum operators in any context, including, here, the harmonic oscillator.

QM B3

QM hand 2:

(a) $\langle x\rangle = \int_{-\infty}^{\infty} x|\psi(x)|^2 dx = 0$ (odd integrand)

$\langle x^2\rangle = \left(\frac{a}{\pi}\right)^{1/2}\int_{-\infty}^{\infty} x^2 e^{-ax^2}dx = \left(\frac{a}{\pi}\right)^{1/2} a^{-3/2}\int_{-\infty}^{\infty} s^2 e^{-s^2}ds$

$s = a^{1/2}x \;\Big|\; = \frac{1}{a\pi^{1/2}}\frac{\pi^{1/2}}{2} = \frac{1}{2a}$

(b) $\phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(x)e^{ikx}dx, \quad k = \frac{p}{\hbar}$

$\phi(k) = \frac{1}{\sqrt{2\pi}}\left(\frac{a}{\pi}\right)^{1/4}\int_{-\infty}^{\infty} e^{-ax^2/2}e^{ikx}dx$

$-ax^2/2 + ikx = -\frac{a}{2}\left(x^2 - \frac{2ikx}{a} - \frac{k^2}{a^2}\right) - \frac{k^2}{2a}$

$\phi(k) = \frac{1}{\sqrt{2\pi}}\left(\frac{a}{\pi}\right)^{1/4} e^{-\frac{k^2}{2a}}\int_{-\infty}^{\infty} e^{-\frac{a}{2}(x-ik/a)^2}dx$

$= \frac{1}{\sqrt{2\pi}}\left(\frac{a}{\pi}\right)^{1/4} e^{-k^2/2a}\left(\frac{2}{a}\right)^{1/2}\pi^{1/2} = \frac{1}{(a\pi)^{1/4}}e^{-\frac{k^2}{2a}}$

(c) $\langle p\rangle = \hbar\langle k\rangle = 0$

$\langle p^2\rangle = \hbar^2\langle k^2\rangle = \frac{\hbar^2}{(a\pi)^{1/2}}\int_{-\infty}^{\infty} k^2 e^{-\frac{k^2}{a}}dx$

$= \frac{\hbar^2}{(a\pi)^{1/2}} a^{3/2}\frac{\pi^{1/2}}{2} = \frac{\hbar^2 a}{2}$

(d) $\Delta x = \sqrt{\langle x^2\rangle} = \frac{1}{\sqrt{2a}} \qquad \Delta p = \hbar\sqrt{\frac{a}{2}}$

$\Delta x\cdot\Delta p = \frac{\hbar}{2}$ in agreement with Heisenberg

$\boxed{\textbf{B4}}$ Consider a two-state quantum system. In the orthonormal and complete set of basis kets $|1\rangle$ and $|2\rangle$, the Hamiltonian operator for the system is represented by ($\omega > 0$):

$$\hat{H} = 10\hbar\omega |1\rangle\langle 1| - 3\hbar\omega |1\rangle\langle 2| - 3\hbar\omega |2\rangle\langle 1| + 2\hbar\omega |2\rangle\langle 2| .$$

Let us consider another orthonormal and complete basis, $|\alpha\rangle$ and $|\beta\rangle$, such that $\hat{H}|\alpha\rangle = E_1|\alpha\rangle$ and $\hat{H}|\beta\rangle = E_2|\beta\rangle$ (with $E_1 < E_2$). Let the action of some operator $\hat{A}$ on the basis kets $|\alpha\rangle$ and $|\beta\rangle$ be given by

$$\hat{A}|\alpha\rangle = 2ia|\beta\rangle \quad \text{and} \quad \hat{A}|\beta\rangle = -2ia|\alpha\rangle - 3a|\beta\rangle ,$$

where a is real and $a > 0$.

a. Show that $\hat{A}$ is Hermitian, and find its eigenvalues.

Answer the next two *independent* parts based on the information given above:

PART I - Suppose an $\hat{A}$-measurement is carried out at time $t = 0$ on an arbitrary state, and the largest possible value is obtained.

b. Calculate the probability $P(t)$ that another measurement made at some later time t will yield the same value as the one measured at $t = 0$.

c. Calculate the time dependence of the expectation value $\langle\hat{A}\rangle$. Plot $\langle\hat{A}\rangle(t)$ as a function of time. What is the minimum value of $\langle\hat{A}\rangle$? At what time is it first achieved?

PART II - Suppose that the average value obtained from a large number of $\hat{A}$-measurements on identical quantum states at a given time is $-a/4$.

d. Construct the most general normalized ket (just before the $\hat{A}$-measurement) for the system consistent with this information. Express your answer as $C|\alpha\rangle + D|\beta\rangle$.

Related to QM B4 -- full solutions elsewhere

$$\hat{H} = 10\hbar\omega|1\rangle\langle 1| - 3\hbar\omega|1\rangle\langle 2| - 3\hbar\omega|2\rangle\langle 1| + 2\hbar\omega|2\rangle\langle 2| = \hbar\omega\begin{pmatrix} 10 & -3 \\ -3 & 2 \end{pmatrix}$$

$$\begin{vmatrix} 10-\lambda & -3 \\ -3 & 2-\lambda \end{vmatrix} = 0 \quad\Rightarrow\quad (10-\lambda)(2-\lambda)-9=0 \quad\Rightarrow\quad 10(2-\lambda)-\lambda(2-\lambda)-9=0 \quad\Rightarrow$$

$$20-10\lambda+\lambda^2-2\lambda-9=0 \quad\Rightarrow\quad \lambda^2-12\lambda+11=0$$

$$D = b^2-4ac = (-12)^2-4\cdot 1\cdot 11 = 144-44 = 100$$

.

$$\lambda = \frac{-b\pm\sqrt{D}}{2a} = \frac{12\pm 10}{2} = 6\pm 5 = 1 \text{ or } 11$$

$$E_1 = \hbar\omega$$

$$E_2 = 11\hbar\omega$$

$\hat{A}|\alpha\rangle = 2ia|\beta\rangle$ and $\hat{A}|\beta\rangle = -2ia|\alpha\rangle - 3a|\beta\rangle$, so

$$\hat{A} = \begin{pmatrix} \langle\alpha|\hat{A}|\alpha\rangle & \langle\alpha|\hat{A}|\beta\rangle \\ \langle\beta|\hat{A}|\alpha\rangle & \langle\beta|\hat{A}|\beta\rangle \end{pmatrix}$$

From $\hat{A}|\alpha\rangle = 2ia|\beta\rangle$ and $\hat{A}|\beta\rangle = -2ia|\alpha\rangle - 3a|\beta\rangle$ we find

$$\langle\alpha|\hat{A}|\alpha\rangle = \langle\alpha|2ia|\beta\rangle = 0$$

$$\langle\alpha|\hat{A}|\beta\rangle = \langle\alpha|-2ia|\alpha\rangle - \langle\alpha|3a|\beta\rangle = -2ia$$

$$\langle\beta|\hat{A}|\alpha\rangle = \langle\beta|2ia|\beta\rangle = 2ia$$

$$\langle\beta|\hat{A}|\beta\rangle = -\langle\beta|2ia|\alpha\rangle - \langle\beta|3a|\beta\rangle = -3a$$

We find $\hat{A} = \begin{pmatrix} 0 & -2ia \\ 2ia & -3a \end{pmatrix}$. We note that, for this matrix, $\hat{A}^\dagger = (\hat{A}^T)^* = \hat{A}$, so $\hat{A}$ is Hermitian.

Eigenvalues:

$$\begin{vmatrix} 0-\lambda & -2ia \\ 2ia & -3a-\lambda \end{vmatrix} = 0 = -\lambda(-3a-\lambda)-4a^2 = \lambda^2+3a\lambda-4a^2$$

$$\lambda^2+3a\lambda-4a^2$$

$$D = B^2-4AC = 9a^2-4*1*(-4a^2) = 25a^2$$

$$\lambda = \frac{-3a\pm\sqrt{D}}{2} = \frac{-3a\pm 5a}{2} = a \quad\text{or}\quad -4a$$

Q.2) In the $|1\rangle, |2\rangle$ basis, H is represented by the matrix $H = \begin{pmatrix} \; & \; \\ -3\hbar\omega & 2\hbar\omega \end{pmatrix}$

QM B4

$|\alpha\rangle$ & $|\beta\rangle$ are eigenvectors of $\hat{H}$ with E_1, E_2 eigenvalues $(E_1 < E_2)$

Find them $\Rightarrow$ $\det \begin{pmatrix} 10\hbar\omega - \lambda & -3\hbar\omega \\ -3\hbar\omega & 2\hbar\omega - \lambda \end{pmatrix} = 0 \Rightarrow \lambda^2 - 12\hbar\omega\lambda + 11\hbar^2\omega^2 = 0 \Rightarrow \lambda_1 = E_1 = \hbar\omega$

$\lambda_2 = E_2 = 11\hbar\omega$

$$\hat{H}|\alpha\rangle = \hbar\omega|\alpha\rangle \Rightarrow |\alpha\rangle = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \frac{1}{\sqrt{10}}|1\rangle + \frac{3}{\sqrt{10}}|2\rangle$$

$$|\beta\rangle = \frac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \frac{3}{\sqrt{10}}|1\rangle - \frac{1}{\sqrt{10}}|2\rangle$$

In the $|\alpha\rangle, |\beta\rangle$ basis, $\hat{A}$ is represented by the matrix $A = \begin{pmatrix} 0 & -2ia_0 \\ 2ia_0 & -3a_0 \end{pmatrix}$

The eigenvalues of $\hat{A}$ are $\det \begin{vmatrix} -\lambda & -2ia_0 \\ 2ia_0 & -\lambda - 3a_0 \end{vmatrix} = 0 \Rightarrow \lambda_1 = a_0 \rightarrow \frac{2}{\sqrt{5}}|\alpha\rangle + \frac{i}{\sqrt{5}}|\beta\rangle$

$\lambda_2 = -4a_0 \rightarrow \frac{1}{\sqrt{5}}|\alpha\rangle - \frac{2i}{\sqrt{5}}|\beta\rangle$

Eigenvalues

Eigenvectors expre in $|\alpha\rangle, |\beta\rangle$ basis

PART I:

a) $\hat{A}$-measurement yielding the largest possible value (must have found a_0, since $a_0 > 0$) collapses $|\psi(0)\rangle$ to the eigenstate of $\hat{A}$ with eigenvalue a_0 (Reduction & measurement postulates)

$$\Rightarrow |\psi(0)\rangle = \frac{2}{\sqrt{5}}|\alpha\rangle + \frac{i}{\sqrt{5}}|\beta\rangle$$

Since $|\psi(0)\rangle$ has already been expressed in terms of $\hat{H}$-eigenstates, it is trivia to write its time evolution,

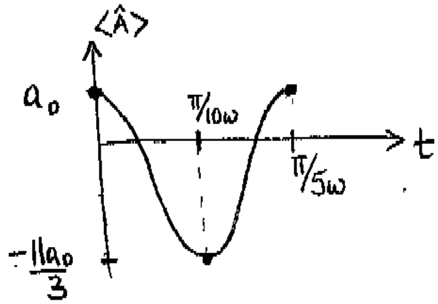
$$|\psi(t)\rangle = \frac{2}{\sqrt{5}}|\alpha\rangle e^{-i\omega t} + \frac{i}{\sqrt{5}}|\beta\rangle e^{-i11\omega t}$$

$\mathbb{P}$ of measuring a_0 again means calculating $|\langle\psi(0)|\psi(t)\rangle|$

$$\mathbb{P}(t) = \left| \underbrace{\begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{-i}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{5}} e^{-i\omega t} \\ \frac{i}{\sqrt{5}} e^{-i11\omega t} \end{pmatrix}}_{\text{all expressed wrt } |\alpha\rangle, |\beta\rangle \text{ basis}} \right|^2 = \left| \frac{4}{5} e^{-i\omega t} + \frac{1}{5} e^{-i11\omega t} \right|^2 = \underline{\underline{\frac{17}{25} + \frac{8}{25}\cos 10\omega t}}$$

(b)
$$\langle\hat{A}\rangle(t) = \underbrace{\begin{pmatrix} \frac{2}{\sqrt{5}} e^{i\omega t} & \frac{-i}{\sqrt{5}} e^{+i11\omega t} \end{pmatrix}}_{\langle\psi(t)|} \underbrace{a_0 \begin{pmatrix} 0 & -2i \\ 2i & -3 \end{pmatrix}}_{\hat{A}} \underbrace{\begin{pmatrix} \frac{2}{\sqrt{5}} e^{-i\omega t} \\ \frac{i}{\sqrt{5}} e^{-i11\omega t} \end{pmatrix}}_{|\psi(t)\rangle} = \frac{a_0}{5} \begin{pmatrix} 2e^{i\omega t} & -ie^{i11\omega t} \end{pmatrix} \begin{pmatrix} 2e^{-i11\omega t} \\ 4ie^{-i\omega t} - 3ie^{-i11\omega t} \end{pmatrix}$$

$$= \frac{a_0}{5}\left(4e^{-i10\omega t} + 4e^{i10\omega t} - 3\right) = \frac{(8\cos 10\omega t - 3)a_0}{5}$$



PART II: Let the probability of obtaining a_0 be $|c_1|^2 \Rightarrow$ probability of obtaining $-4a_0$ will be $(1-|c_1|^2) \Rightarrow$

$$\langle A\rangle = |c_1|^2 a_0 + (1-|c_1|^2)(-4a_0) = -\frac{a_0}{4} \Rightarrow |c_1|^2 = \frac{3}{4}$$

$\Rightarrow |\psi\rangle = \frac{\sqrt{3}}{2}|\gamma\rangle + \frac{e^{i\delta}}{2}|\delta\rangle$ where δ is an arbitrary phase factor and $|\gamma\rangle$ and $|\delta\rangle$ are eigenvectors of $\hat{A}$ with eigenvalues a_0 & $-4a_0$, respectively.

From diagonalization of $\hat{A}$ before, we have $|\gamma\rangle = \frac{2}{\sqrt{5}}|\alpha\rangle + \frac{i}{\sqrt{5}}|\beta\rangle$

and $|\delta\rangle = \frac{1}{\sqrt{5}}|\alpha\rangle - \frac{2i}{\sqrt{5}}|\beta\rangle$

$$\Rightarrow |\psi\rangle = \frac{\sqrt{3}}{2}\left(\frac{2}{\sqrt{5}}|\alpha\rangle + \frac{i}{\sqrt{5}}|\beta\rangle\right) + \frac{e^{i\delta}}{2}\left(\frac{1}{\sqrt{5}}|\alpha\rangle - \frac{2i}{\sqrt{5}}|\beta\rangle\right) = \underbrace{\left(\sqrt{\frac{3}{5}} + \frac{e^{i\delta}}{2\sqrt{5}}\right)}_{C}|\alpha\rangle + \underbrace{\left(\frac{i\sqrt{3}}{2\sqrt{5}} - \frac{ie^{i\delta}}{\sqrt{5}}\right)}_{D}|\beta\rangle$$