UNL - Department of Physics and Astronomy

Preliminary Examination - Day 1
Thursday, August 9, 2018

This test covers the topics of *Thermodynamics and Statistical Mechanics* (Topic 1) and *Quantum Mechanics* (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

Note: If you do more than two problems in a group, only the first two (in the order they appear in this handout) will be graded. For instance, if you do problems A1, A3, and A4, only A1 and A3 will be graded.

WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY

Thermodynamics and Statistical Mechanics Group A - Answer only two Group A questions

A1 Derive the expression for the efficiency, defined as the total work done over the total heat supplied, for a Carnot cycle which uses a monoatomic ideal gas as an operating substance. Use the equation of state for the gas $PV = nRT$ and the internal energy $U = \frac{3}{2}nRT$.

A2 Prove that the C_p for an ideal gas is independent of pressure. Reminder: heat capacity at constant pressure can be defined as $C_p = (\partial H/\partial T)_p$.

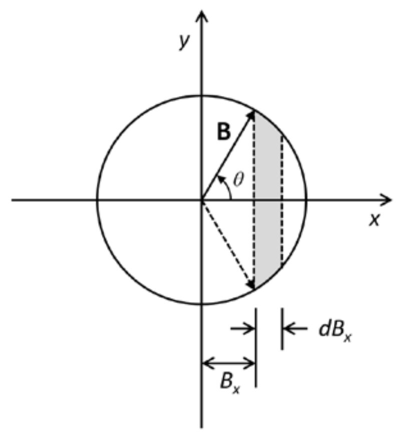
A3 The internal energy for 1 kg of a certain gas, in joules, is given by $U = 0.17\,T + C$ where T is the gas temperature in kelvin, and C is a constant. The gas is heated in a rigid container (i.e. at constant volume) from a temperature of 40°C to 316°C. Compute the amount of work and heat flow into the system.

A4 A large number of non-interacting particles is in equilibrium with a thermal bath of temperature 300 K. The particles have only three energy levels: $E_1 = 20$ meV, $E_2 = 30$ meV, and $E_3 = 40$ meV. Calculate the average energy of a particle.

<u>Thermodynamics and Statistical Mechanics Group B</u> <u>- Answer only two Group B questions</u>

B1 Consider mixing 100 g of water at 300 K with 50 g of water at 400 K. Calculate the final equilibrium temperature if the specific heat c of water per gram is 1 cal/g/K. Calculate the change in entropy for this irreversible process.

B2 A two-dimensional vector $\mathbf{B}$ of constant length $B = |\mathbf{B}|$ is equally likely to point in any direction specified by the angle θ. What is the probability that the x-component of this vector lies between B_x and $B_x + dB_x$?



B3 Show that the work done by a gas under arbitrary changes of temperature and pressure can be determined in terms of the coefficient of volume expansion at constant pressure α_p and the isothermal compressibility coefficient κ_T. As a corollary, show that for an isochoric (constant volume) process

$$\left(\frac{\partial p}{\partial T}\right)_V = \frac{\alpha_p}{\kappa_T}$$

Verify this for an ideal gas. Reminder: the involved coefficients are defined as

$\alpha_p = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p$ and $\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T$.

B4 Consider a paramagnetic material whose magnetic particles have angular momentum quantum number J which is an odd multiple of ½. The z-component can take $2J+1$ values ($J_z = -J, -J+1, -J+2, \ldots, J$). This leads to $2J+1$ allowed values of the z-component of a particle's magnetic moment: $\mu_z = -J\mu_0, (-J+1)\mu_0, (-J+2)\mu_0, \ldots, J\mu_0$, where μ_0 is some unit magnetic moment*. The energy of the magnetic moment in a magnetic field pointing in the +z direction is $-\mu_z B$.

**Not to be confused with the magnetic permeability for which the same symbol is used!*

a. Derive an expression for the partition function Z_1 of a single magnetic particle in a magnetic field B pointing in the +z direction. Write your answer in terms of hyperbolic sine functions. You may find it convenient to use the variable $b = \mu_0 B\beta$, where $\beta = 1/k_B T$ as usual.

b. Derive an expression for the average energy of the particle in part (*a*). Give your answer in terms of the hyperbolic cotangent function.

Quantum Mechanics Group A - Answer only two Group A questions

A1 Let α and β be eigenstates of the electron spin operator S_z with the eigenvalues $\hbar/2$ and $-\hbar/2$ respectively. Find

$$S_+\alpha, \quad S_+\beta, \quad S_-\alpha, \quad S_-\beta, \quad S_+S_-\alpha,$$

where $S_\pm = S_x \pm iS_y$.

A2 A particle of mass m is moving in the one-dimensional potential

$$V(x) = -V_0\, e^{-(x/a)^2}$$

where V_0 and a are positive constants.

a. Consider an energy eigenstate with $E < 0$. Is it a parity eigenstate? If yes, what are possible parity eigenvalues?

b. Consider now a state with $E > 0$ corresponding to the particle incident from $-\infty$. Write down the general form of the particle's wavefunction at $x \ll -a$ and $x \gg a$. Don't calculate constant coefficients in your expressions, but explain their meaning.
Is this state a parity eigenstate? If yes, what are possible parity eigenvalues?
Is this state a momentum eigenstate?

A3 A wavefunction in one dimension is given by

$$\psi(x) = \begin{cases} -C & \text{for } -a < x < 3a \\ 0 & \text{elsewhere} \end{cases}$$

where C and a are positive constants. Calculate the expectation value of the parity operator.

A4 The spherical harmonics are orthonormal; we have

$$\oint Y^*_{\ell,m}(\theta,\phi) Y_{\ell',m'}(\theta,\phi)\, d\Omega = \delta_{\ell\ell'}\delta_{mm'}$$

where $d\Omega$ is an infinitesimal amount of solid angle, and the integral is taken over all solid angle. Use this expression to demonstrate that $Y_{1,0}$ and $Y_{1,1}$ are orthogonal.

Quantum Mechanics Group B - Answer only two Group B questions

B1 A 200-keV photon collides with an electron at rest. The photon is scattered at 90°.

a. What is the photon energy after the collision?

b. What is the kinetic energy of the electron after the collision?

c. What is the angle between the photon's momentum and the electron's momentum after the collision?

d. How accurate would the result for the angle be if the electron were treated nonrelativistically?

B2 *NOTE: In this problem, we encounter infinitely large matrices, We will write these by only specifying the 4 by 4 block in the upper left corner, as in* $\begin{pmatrix} ? & ? & ? & ? & \cdots \\ ? & ? & ? & ? & \\ ? & ? & ? & ? & \\ ? & ? & ? & ? & \\ \vdots & & & & \ddots \end{pmatrix}$. *For instance, the identity operator is written as* $\hat{I} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \\ \vdots & & & & \ddots \end{pmatrix}$.

The stationary states of the harmonic oscillator are defined by $\hat{H}\,|\,n\rangle = (n+\frac{1}{2})\hbar\omega\,|\,n\rangle$.

The annihilation operator $\hat{a}$ of the harmonic oscillator is defined by $\hat{a} = \frac{\beta}{\sqrt{2}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)$ (with $\beta^2 = m\omega/\hbar$). The operation of the annihilation operator is $\hat{a}\,|\,n\rangle = \sqrt{n}\,|\,n-1\rangle$. Thus, in the $|\,n\rangle$ basis, the annihilation operator's matrix is $\hat{a} = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & \\ 0 & 0 & 0 & \sqrt{3} & \\ 0 & 0 & 0 & 0 & \\ \vdots & & & & \ddots \end{pmatrix}$

a. Explain why $\hat{a}^\dagger = \hat{a}^{\mathrm{T}}$, where T means matrix transposition.

b. Find the matrix for $\hat{a}^\dagger$.

c. Find the matrix for $\hat{x}$.

d. Find the matrix for $\hat{p}$.

e. Find the matrix for $\hat{x}\hat{p}$.

f. Explain why $\hat{p}\hat{x} = [(\hat{x}\hat{p})^{\mathrm{T}}]^*$, where T means matrix transposition.

g. Find the matrix for $\hat{p}\hat{x}$.

h. Find the matrix for $[\hat{x},\hat{p}]$ and comment on your answer.

B3 The harmonic oscillator ground-state wavefunction is given by

$$\psi(x) = \left(\frac{a}{\pi}\right)^{1/4} e^{-ax^2/2}$$

where $a = m\omega/\hbar$ (m is mass, and ω is frequency).

(a) Find the expectation values of x and x^2;

(b) Find the wavefunction in the momentum space;

(c) Find the expectation values of p and p^2;

(d) Calculate the uncertainties Δx and Δp and show that they satisfy Heisenberg's uncertainty principle.

B4 Consider a two-state quantum system. In the orthonormal and complete set of basis kets $|1\rangle$ and $|2\rangle$, the Hamiltonian operator for the system is represented by ($\omega > 0$):

$$\hat{H} = 10\hbar\omega|1\rangle\langle 1| - 3\hbar\omega|1\rangle\langle 2| - 3\hbar\omega|2\rangle\langle 1| + 2\hbar\omega|2\rangle\langle 2|.$$

Let us consider another orthonormal and complete basis, $|\alpha\rangle$ and $|\beta\rangle$, such that $\hat{H}|\alpha\rangle = E_1|\alpha\rangle$ and $\hat{H}|\beta\rangle = E_2|\beta\rangle$ (with $E_1 < E_2$). Let the action of some operator $\hat{A}$ on the basis kets $|\alpha\rangle$ and $|\beta\rangle$ be given by

$$\hat{A}|\alpha\rangle = 2ia|\beta\rangle \quad \text{and} \quad \hat{A}|\beta\rangle = -2ia|\alpha\rangle - 3a|\beta\rangle \ ,$$

where a is real and $a > 0$.

a. Show that $\hat{A}$ is Hermitian, and find its eigenvalues.

Answer the next two *independent* parts based on the information given above:

PART I - Suppose an $\hat{A}$-measurement is carried out at time $t = 0$ on an arbitrary state, and the largest possible value is obtained.

b. Calculate the probability $P(t)$ that another measurement made at some later time t will yield the same value as the one measured at $t = 0$.

c. Calculate the time dependence of the expectation value $\langle\hat{A}\rangle$. Plot $\langle\hat{A}\rangle(t)$ as a function of time. What is the minimum value of $\langle\hat{A}\rangle$? At what time is it first achieved?

PART II - Suppose that the average value obtained from a large number of $\hat{A}$-measurements on identical quantum states at a given time is $-a/4$.

d. Construct the most general normalized ket (just before the $\hat{A}$-measurement) for the system consistent with this information. Express your answer as $C|\alpha\rangle + D|\beta\rangle$.

Physical Constants

speed of light .... $c = 2.998\times10^{8}$ m/s	electrostatic constant ... $k = (4\pi\varepsilon_0)^{-1} = 8.988\times10^{9}$ m/F
Planck's constant .... $h = 6.626\times10^{-34}$ J·s	electron mass .... $m_{\mathrm{el}} = 9.109\times10^{-31}$ kg
Planck's constant / 2π.... $\hbar = 1.055\times10^{-34}$ J·s	electron rest energy...... 511.0 keV
Boltzmann constant ...... $k_{\mathrm{B}} = 1.381\times10^{-23}$ J/K	Compton wavelength .. $\lambda_{\mathrm{C}} = h/m_{\mathrm{el}}c = 2.426$ pm
elementary charge .... $e = 1.602\times10^{-19}$ C	proton mass .... $m_{\mathrm{p}} = 1.673\times10^{-27}\,\mathrm{kg} = 1836 m_{\mathrm{el}}$
electric permittivity ...... $\varepsilon_0 = 8.854\times10^{-12}$ F/m	1 bohr .... $a_0 = \hbar^2/ke^2 m_{\mathrm{el}} = 0.5292$ Å
magnetic permeability ... $\mu_0 = 1.257\times10^{-6}$ H/m	1 hartree (= 2 rydberg) ... $E_{\mathrm{h}} = \hbar^2/m_{\mathrm{el}}{a_0}^2 = 27.21$ eV
molar gas constant.......... $R = 8.314$ J / mol·K	gravitational constant ... $G = 6.674\times10^{-11}$ m³/ kg s²
Avogadro constant .... $N_{\mathrm{A}} = 6.022\times10^{23}$ mol^{-1}	hc .... $hc = 1240$ eV·nm

Equations That May Be Helpful

TRIGONOMETRY

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

$$\sin\alpha\sin\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$
$$\cos\alpha\cos\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$
$$\sin\alpha\cos\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$
$$\cos\alpha\sin\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$$

HYPERBOLIC FUNCTIONS

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$
$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$
$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$
$$\coth(x) = \frac{1}{\tanh(x)}$$

THERMODYNAMICS

Partition function = $Z = \Sigma_i e^{-\beta E_i}$

Average energy = $\langle E \rangle = -\frac{\partial}{\partial \beta}(\ln Z)$

Heat capacity = $C_V = N\frac{d\langle E \rangle}{dT}$

Clausius' theorem: $\sum_{i=1}^{N} \frac{Q_i}{T_i} \le 0$, which becomes $\sum_{i=1}^{N} \frac{Q_i}{T_i} = 0$ for a reversible cyclic process of N steps.

$$\frac{dp}{dT} = \frac{\lambda}{T\Delta V}$$

For adiabatic processes in an ideal gas with constant heat capacity, $pV^{\gamma} = \text{const.}$

$$dU = TdS - pdV$$
$$H = U + pV \qquad F = U - TS \qquad G = F + pV \qquad \Omega = F - \mu N$$

$$C_V = \left(\frac{\delta Q}{dT}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V \qquad C_p = \left(\frac{\delta Q}{dT}\right)_p = T\left(\frac{\partial S}{\partial T}\right)_p \qquad TdS = C_V dT + T\left(\frac{\partial S}{\partial V}\right)_T dV$$

$$\kappa = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T \qquad \alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p$$

Triple product: $\left(\frac{\partial X}{\partial Y}\right)_Z \cdot \left(\frac{\partial Y}{\partial Z}\right)_X \cdot \left(\frac{\partial Z}{\partial X}\right)_Y = -1$

Maxwell's relations:

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial p}{\partial S}\right)_V , \quad \left(\frac{\partial T}{\partial p}\right)_S = \left(\frac{\partial V}{\partial S}\right)_p , \quad \left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V , \quad -\left(\frac{\partial S}{\partial p}\right)_T = \left(\frac{\partial V}{\partial T}\right)_p$$

Data for water

specific heat	$C = 4186\ \mathrm{J/(kg{\cdot}K)}$
heat of fusion	$L_\mathrm{F} = 334\ \mathrm{kJ/kg}$
heat of vaporization	$L_\mathrm{V} = 2256\ \mathrm{kJ/kg}$

QUANTUM MECHANICS

Ground-state wavefunction of the hydrogen atom: $\psi(\mathbf{r}) = \dfrac{e^{-r/a_0}}{\pi^{1/2} a_0^{3/2}}$, where $a_0 = \dfrac{4\pi\varepsilon_0 \hbar^2}{me^2}$ is the Bohr radius, using $m \approx m_\mathrm{el}$, in which m_el is the electron mass.

$$\psi_{nlm}(\mathbf{r}) = R_{nl}(r) Y_{lm}(\hat{\mathbf{r}}) \qquad E_n = -\frac{1}{n^2}\frac{mk^2e^4}{2\hbar^2}$$

$$R_{10}(r) = \frac{2}{a_0^{3/2}} e^{-r/a_0}$$

$$R_{21}(r) = \frac{1}{3^{1/2}(2a_0)^{3/2}} \frac{r}{a_0} e^{-r/2a_0}$$

Particle in one-dimensional, infinitely-deep box with walls at $x = 0$ and $x = a$:

Stationary states $\psi_n = (2/a)^{1/2} \sin(n\pi x/a)$, energy levels $E_n = n^2 \dfrac{\pi^2\hbar^2}{2ma^2}$

Angular momentum: $[L_x, L_y] = i\hbar L_z$ *et cycl.*

Ladder operators:

$$L_+ |\ell, m\rangle = \hbar\sqrt{(\ell+m+1)(\ell-m)}\, |\ell, m+1\rangle$$
$$L_- |\ell, m\rangle = \hbar\sqrt{(\ell+m)(\ell-m+1)}\, |\ell, m-1\rangle$$

Creation, annihilation operators:

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right) \qquad \hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right)$$

$$\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \qquad \hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$$

Probability current density: $J(x) = \frac{\hbar}{2mi}\left(\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right) = \frac{\hbar}{m}\mathrm{Im}\left(\psi^*\frac{\partial\psi}{\partial x}\right).$

Table Spherical harmonics and their expressions in Cartesian coordinates.

$Y_{lm}(\theta,\varphi)$	$Y_{lm}(x,y,z)$
$Y_{00}(\theta,\varphi) = \frac{1}{\sqrt{4\pi}}$	$Y_{00}(x,y,z) = \frac{1}{\sqrt{4\pi}}$
$Y_{10}(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta$	$Y_{10}(x,y,z) = \sqrt{\frac{3}{4\pi}}\frac{z}{r}$
$Y_{1,\pm1}(\theta,\varphi) = \mp\sqrt{\frac{3}{8\pi}}e^{\pm i\varphi}\sin\theta$	$Y_{1,\pm1}(x,y,z) = \mp\sqrt{\frac{3}{8\pi}}\frac{x\pm iy}{r}$
$Y_{20}(\theta,\varphi) = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$	$Y_{20}(x,y,z) = \sqrt{\frac{5}{16\pi}}\frac{3z^2-r^2}{r^2}$
$Y_{2,\pm1}(\theta,\varphi) = \mp\sqrt{\frac{15}{8\pi}}e^{\pm i\varphi}\sin\theta\cos\theta$	$Y_{2,\pm1}(x,y,z) = \mp\sqrt{\frac{15}{8\pi}}\frac{(x\pm iy)z}{r^2}$
$Y_{2,\pm2}(\theta,\varphi) = \sqrt{\frac{15}{32\pi}}e^{\pm 2i\varphi}\sin^2\theta$	$Y_{2,\pm2}(x,y,z) = \mp\sqrt{\frac{15}{32\pi}}\frac{x^2-y^2\pm 2ixy}{r^2}$

$H_{\mathrm{mag}} = -\gamma\,\mathbf{S}\cdot\mathbf{B}$

Pauli matrices: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Compton scattering: $\lambda' - \lambda = \lambda_C(1-\cos\theta)$

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl : $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem : $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

CARTESIAN AND SPHERICAL UNIT VECTORS

$$\hat{\mathbf{x}} = (\sin\theta\cos\phi)\hat{\mathbf{r}} + (\cos\theta\cos\phi)\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{y}} = (\sin\theta\sin\phi)\hat{\mathbf{r}} + (\cos\theta\sin\phi)\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

INTEGRALS

$f(x)$	$\int_0^\infty f(x)\,dx$
e^{-ax^2}	$\dfrac{\sqrt{\pi}}{2\sqrt{a}}$
xe^{-ax^2}	$\dfrac{1}{2a}$
$x^2 e^{-ax^2}$	$\dfrac{\sqrt{\pi}}{4a^{3/2}}$
$x^3 e^{-ax^2}$	$\dfrac{1}{2a^2}$
$x^4 e^{-ax^2}$	$\dfrac{3\sqrt{\pi}}{8a^{5/2}}$
$x^5 e^{-ax^2}$	$\dfrac{1}{a^3}$
$x^6 e^{-ax^2}$	$\dfrac{15\sqrt{\pi}}{16a^{7/2}}$

$$\int_0^\infty \frac{1}{1+bx^2}\,dx = \pi / 2b^{1/2}$$

$$\int_0^\infty x^n e^{-bx}\,dx = \frac{n!}{b^{n+1}}$$

$$\int (x^2+b^2)^{-1/2}\,dx = \ln\left(x+\sqrt{x^2+b^2}\right)$$

$$\int (x^2+b^2)^{-1}\,dx = \frac{1}{b}\arctan(x/b)$$

$$\int (x^2+b^2)^{-3/2}\,dx = \frac{x}{b^2\sqrt{x^2+b^2}}$$

$$\int (x^2+b^2)^{-2}\,dx = \frac{\dfrac{bx}{x^2+b^2}+\arctan(x/b)}{2b^3}$$

$$\int \frac{x\,dx}{x^2+b^2} = \frac{1}{2}\ln\left(x^2+b^2\right)$$

$$\int \frac{dx}{x(x^2+b^2)} = \frac{1}{2b^2}\ln\left(\frac{x^2}{x^2+b^2}\right)$$

$$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right) = -\frac{1}{ab}\operatorname{artanh}\left(\frac{ax}{b}\right)$$