UNL Department of Physics and Astronomy

Preliminary Examination – Day 1

August, 2011

This test covers the topics of Mechanics (Topic 1) and Thermodynamics and Statistical Mechanics (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work 2 problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

Mechanics Group A. Answer only 2 Group A questions.

A1. The trajectory of a body of mass m, measured in a frame A, is given by

$$\boldsymbol{r} = (v_o t + x_o sin\omega t)\hat{\boldsymbol{x}} + (\frac{1}{2}gt^2 + y_o cos\omega t)\hat{\boldsymbol{y}} \ ,$$

where v_0, g, ω, x_0, and y_0 are constants. The force applied to the body is known to be

$$\boldsymbol{F} = \frac{1}{2}(x_o m\omega^2 sin\omega t)\hat{\boldsymbol{x}} + \left(mg + \frac{1}{2}y_o m\omega^2 cos\omega t\right)\hat{\boldsymbol{y}}.$$

Is the frame A inertial? Explain.

A2. A mass m is attached to a spring and is oscillating as x = Acos(ωt + ϕ). A very short impulse is to be applied to the mass to stop the motion. What is the magnitude of the impulse? When must it be applied?

A3. Consider two cylindrical bodies (A and B) of equal mass and dimensions with moments of inertia about the cylindrical axis of I_A and I_B, respectively. For both bodies, the center of mass coincides with the geometric center. Each is released from rest and allowed to roll down an inclined ramp. Assuming the center of mass of each body is initially at the same location, which body will have the greater center of mass velocity at the bottom of the ramp. Why?

A4. A particle of mass m moves in a potential $V(x) = \beta x^4 - \alpha x^2$, where α and β are constants and are both greater than zero. Find the equilibria and determine their stability. Find the frequency of small oscillations of a mass about the stable equilibria. What are the dimensions of α and β ?

Mechanics Group B. Answer only 2 Group B questions.

B1. A satellite is in a circular orbit of radius r_1 about the center of the Earth. A short thrust of the satellite's engine is used to reduce the velocity by a factor s without changing its direction. This puts the satellite into an elliptical orbit having radius r_1 at apogee (farthest distance) and radius r_2 at perigee (closest distance). Find the ratio r_1/r_2 as a function of s.

B2. For the Lagrangian

$$L = \frac{m}{2}[\dot{q}_1^2(1 + sin^2 q_2) + \dot{q}_2^2] - \frac{k}{3}q_2^3$$

where k and m are constants:

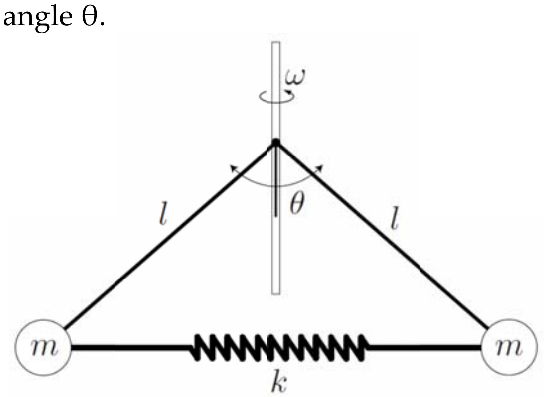
(a) Find the equations of motion. (Don't try to solve the equations.)

(b) Are there any ignorable (i.e. cyclic) coordinates? If so, what are the corresponding conserved momenta?

(c) Find the energy of the system. Is it conserved? Why?

B3. A block of mass m is released from the top of a fixed wedge. The face of the wedge is inclined at an angle θ to the horizontal. Initially, the coefficient of friction between block and the wedge is μ. As the surface of the block heats up, μ decreases linearly with the energy absorbed. Find the coefficient of friction as a function of the distance traveled by the block.

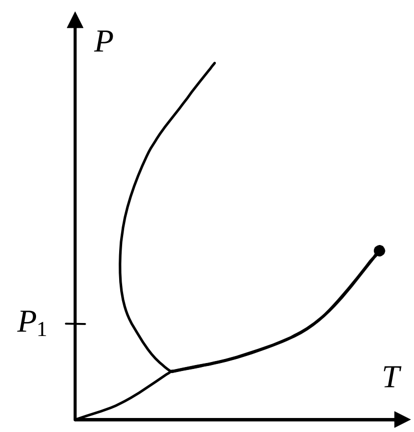
B4. A pair of masses is attached to a pivot on a vertical shaft by massless rods of length ℓ and can swing in the plane as shown. The shaft is rotating with a constant angular frequency ω. The masses are also connected to each other by a spring as shown. Find the equilibrium value of the angle θ.



Thermodynamics and Statistics Group A. Answer only 2 Group A questions.

A1. A die is thrown N times, and the ***sum*** of all N shown numbers is recorded. Find the expectation value and the standard deviation for this sum.

A2. The figure shows the phase diagram for a one-component substance. Put labels on the graph to indicate the areas of the solid (S), liquid (L), and gas (G) phase. If the solid phase is in equilibrium with the liquid phase at pressure P_1, will the solid float or sink? Is there a pressure at which the solid will neither float nor sink? Explain your reasoning.



A3. Consider a gas characterized by an unknown equation of state P=P(V,T). Express $\left(\frac{\partial P}{\partial T}\right)_V$ in terms of $\left(\frac{\partial P}{\partial V}\right)_T$ and $\left(\frac{\partial V}{\partial T}\right)_P$.

A4. Write down briefly but precisely

A)The Kelvin statement of the second law.

B) The Clausius statement of the second law.

C) The entropy statement of the second law.

Thermodynamics and Statistics Group B. Answer only 2 Group B questions.

B1. Two copper blocks (implying constant heat capacity) at temperature T_h and T_c, where $T_c < T_h$, are allowed to equilibrate by means of a reversible heat engine operating between the two blocks.

a) What is the final temperature of both blocks (when the heat engine stops producing work) in terms of T_h and T_c?

b) What is the total change in entropy of the two blocks?

B2. A reversible heat engine extracts heat Q_h from a reservoir at temperature T_h and heat $Q_m = aQ_h$, with $0 < a < 1$, from a reservoir at temperature $T_m < T_h$ while rejecting heat Q_c to a reservoir at temperature $T_c < T_m$.

Derive an expression for the efficiency of this three-reservoir heat engine in terms of a and the three reservoir temperatures T_h, T_m, and T_c. The efficiency is given by the produced work divided by the heat extracted from the two hotter reservoirs. Check your result by showing that the efficiency reduces to the Carnot efficiency expression in the limits $a \rightarrow 0$ and $T_m \rightarrow T_c$.

B3. 500 grams of ice cubes at 0°C are placed in 1 liter of water at 20°C. The system then comes to equilibrium with no heat exchange with the surroundings.

(a) Does the ice melt completely? If yes, find the temperature of the water in equilibrium. If not, find how much ice remains in equilibrium.

(b) Calculate the total change of entropy for the whole system.

B4. A certain substance in a vessel undergoes the following process:

(a) It expands isothermally while receiving 1000 J of heat from the thermostat at 1000 K.
(b) It is insulated and then expanded somewhat.
(c) It is compressed isothermally while releasing 250 J to the second thermostat at 500 K.
(d) It is insulated and then expanded somewhat.
(e) It is compressed isothermally while releasing some heat to the third thermostat at 300 K.
(f) It is insulated and then compressed in such a way that it ends up in the initial state.

Given these restrictions, what is the maximum amount of work that can be done by the system in such a process?

EQUATIONS THAT MAY BE HELPFUL

Elliptic Orbits

$$r = \frac{m\ell^2}{k}\frac{1}{1+\epsilon\cos(\theta-\theta_0)}$$

$$\frac{r_2}{r_1} = \frac{1-\epsilon}{1+\epsilon}$$

$$\epsilon = \sqrt{1+\frac{2Em\ell^2}{k^2}}$$

$$r_1 + r_2 = -\frac{k}{E}$$

Rigid Body motion

Body	Axis	I
Sphere; radius r	Any axis	$\frac{2}{5}m\,r^2$
Cube; side a	Any axis	$\frac{1}{6}m\,a^2$
Rod; radius a, length l	Along the cylindrical axis	$\frac{1}{2}m\,a^2$
	Through center, perpendicular to the cylindrical axis	$\frac{1}{4}m\,a^2 + \frac{1}{12}m\,l^2$

$$T = \frac{1}{2}I\,\omega^2$$

$$L = I\,\omega$$

$$\boldsymbol{a}' = \boldsymbol{a} - 2\,\boldsymbol{\omega}\times\boldsymbol{v}' - \dot{\boldsymbol{\omega}}\times\boldsymbol{r}' - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}')$$

General efficiency η of a heat engine producing work $|W|$ while taking in heat $|Q_h|$ is $\eta = \frac{|W|}{|Q_h|}$.

For a Carnot cycle operating as a heat engine between reservoirs at T_h and at T_c the efficiency becomes $\eta_C = \frac{T_h - T_c}{T_h}$.

Clausius' theorem: $\sum_{i=1}^{N}\frac{Q_i}{T_i} \le 0$ which become $\sum_{i=1}^{N}\frac{Q_i}{T_i} = 0$ for a reversible cyclic process of N steps.

$\frac{dP}{dT} = \lambda/(T\Delta V)$; specific heat of water: 4186 J/(kg*K); Latent heat of ice melting: 334 J/g

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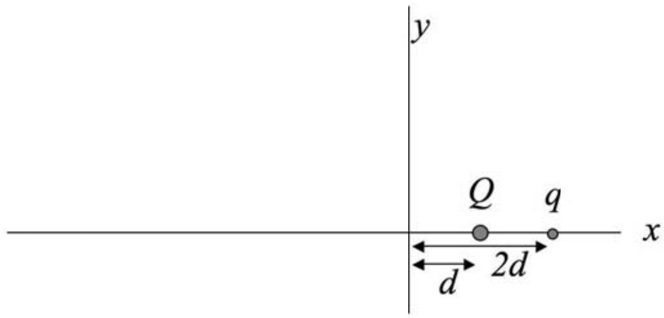
Preliminary Examination – Day 2

August, 2011

This test covers the topics of Electricity and Magnetism (Topic 1) and Quantum Mechanics (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work 2 problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

Electricity and Magnetism Group A. Answer only 2 Group A questions.

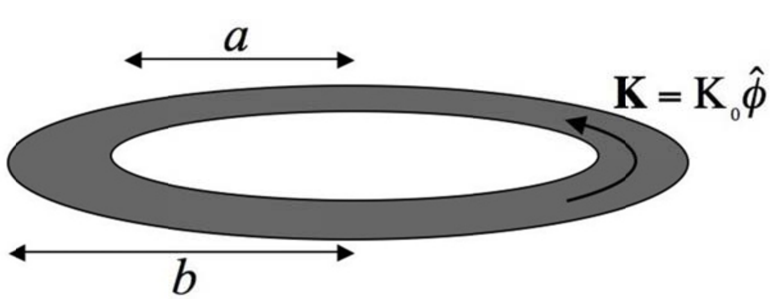
A1. A positive charge q is placed on the x-axis at $x = 2d$ and a second charge $Q = -2q$ at $x = d$. How does the net force acting on the positive charge compare to the case where an infinite, grounded plane conductor is placed in the yz-plane? Is it in the same direction? Larger, smaller, or just the same? Find the ratio of the two forces.



A2. Consider a region of space containing an E field. A Linear Isotropic Homogeneous dielectric is then brought into this region of space and the presence of the dielectric modifies the E field. Does the electrostatic energy U_E increase, decrease or stay the same? Why?

A3. A charged particle with zero initial velocity enters a region containing both an E and B field. The E field is given by $\vec{E} = E_0\hat{i}$ and the particle moves in a trajectory that exactly follows the E field. What is the direction of the B field and what is the sign of the particle's charge? (In this problem $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along the x,y, and z direction respectively.)

A4. What is the magnetic dipole moment of a flat circular disc with inner radius a, outer radius b, and surface current density **K**=$K_0\hat{\varphi}$**?** Hint: the dipole moment of a infinitesimally thin ring of radius r carrying a current I is $\pi r^2 I$.



Electricity and Magnetism Group B. Answer only 2 Group B questions.

B1. Consider an infinitely long solid cylinder of radius R along the z axis. A current flows up the wire with a volume current density given by $\vec{J} = J_0\hat{z}$.

(a) Draw a figure indicating coordinate axes, the cylinder and the direction of current flow. Explain clearly the limitations that the symmetry of this structure imposes on the B field as well as the limitations resulting from the Biot-Savart Law.

(b) Using Ampere's law calculate the B field vector both inside and outside the cylinder. Draw and describe the Amperian loop to be used in each case.

B2. Consider two dipoles $\mathbf{m}_1$ and $\mathbf{m}_2$ a distance d apart oriented so that both are perpendicular to d as shown. $\mathbf{m}_1$ is fixed and $\mathbf{m}_2$ is free to move in any fashion.

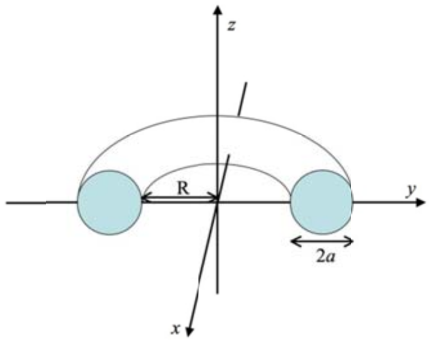
(a) Re-draw the diagram above and indicate an appropriate choice of coordinate system.

(b) What is the torque exerted by dipole $\mathbf{m}_1$ on dipole $\mathbf{m}_2$? Describe the resultant motion of $\mathbf{m}_2$ under the action of this torque.

(c) What is the force exerted by dipole $\mathbf{m}_1$ on dipole $\mathbf{m}_2$? Describe the resultant motion of $\mathbf{m}_2$ under the action of this force.

Electricity and Magnetism Group B (continued). Answer only 2 Group B questions.

B3. A cylinder of length L and radius a has a uniform magnetization M_0 directed along its axis. The cylinder is then bent into a torus of inner radius R, without changing its magnetization. The center of the torus is at the origin and the central plane of the torus lies in the x-y plane. A cut away view is shown below.



(a) Find the magnitude and direction of the bound volume current density, $\mathbf{J}_b$. Indicate the direction on the diagram (which you should redraw), on both the right and left cross sectional views.

(b) Find the magnitude and direction of the bound surface current density, $\mathbf{K}_b$. Indicate the direction on the diagram, on both the right and left cross sectional views.

(c) The **B** field everywhere inside the "hole" of the torus is zero. Using the appropriate boundary conditions, calculate the magnitude and direction of the **B** field at a point P just within the torus and on the y axis i.e. at a coordinate point (0, R+δ, 0) where δ is very small. Note: There are two tangential directions-be careful which one you choose.

B4. The electric field in a region **r > R** is given by $E_r = \dfrac{2A\cos\theta}{r^3}$, $E_\theta = \dfrac{A\sin\theta}{r^3}$ and $E_\phi = 0$, where these refer to the components of the E field in a spherical coordinate system and where A is a positive constant.

(a) Draw a figure of a sphere of radius r = R, including appropriate axes, and the *direction and magnitude* of the E field at three points at the surface of the sphere, i.e. with r=R: at the North pole, at the South pole, and at the Equator.

(b) Calculate the charge density in the region r > R. Show all work.

(c) From your results in (ii) and from the E field given in the problem, what can you say about the volume charge density? Where do the charges exist? Is the charge density isotropic? Justify your answers.

Quantum Mechanics Group A. Answer only 2 Group A questions.

A1. What is the kinetic energy of an electron that has the same momentum as an 80-keV photon?

A2. A particle with mass m is described by the normalized wavefunction

$$\psi(x) = \begin{cases} C\cos(\tfrac{1}{2}\pi x/a) & \text{for } -a < x < a \\ 0 & \text{elsewhere} \end{cases},$$

Where a and C are positive real-valued constants. If we measure the particle's momentum, what is the probability that we find it moving to the left? Explain your answer.

A3. This problem refers to motion in one dimension along the x-axis. Is $[x^2, p_x^2]$ Hermitian? Explain.

A4. A three-dimensional Hilbert space $\mathcal{H}_3$ is spanned by the complete set of orthonormal kets: $|u_1\rangle, |u_2\rangle$, and $|u_3\rangle$. At some point in time, the quantum system is in the state

$$|\psi\rangle = -\tfrac{1}{2}e^{\frac{1}{3}\pi i}\sqrt{3}\,|u_1\rangle + \tfrac{1}{2}|u_3\rangle.$$

(a) Show that $|\psi\rangle$ is normalized.

At the same point in time, we do a measurement of an observable S with the associated operator S. We find the non-degenerate value $S = s$, which corresponds to the eigenket

$$|s\rangle = -\tfrac{5}{13}i\,|u_2\rangle - \tfrac{12}{13}|u_3\rangle.$$

(b) What was the probability to find this value s?

Quantum Mechanics Group B. Answer only 2 Group B questions.

B1. A free particle of mass m, moving in a one-dimensional space, has the following normalized wavefunction in momentum space (momentum probability amplitude):

$$\langle p | \psi \rangle = \varphi(p) = \begin{cases} K & \text{for } |p| \le \dfrac{h}{2a} \\ 0 & \text{for all other } p \end{cases} \qquad (K \text{ is a real-valued constant})$$

The eigenkets $| p \rangle$ of the momentum operator p satisfy completeness: $\int_{-\infty}^{\infty} dp \, | p \rangle \langle p | = 1$.

(a) Find the constant *K*, including units.

(b) Find the wavefunction $\psi(x)$ in position space.

B2. A point particle with mass m moves in one dimension (coordinate x) in the potential

$$V(x) = \tfrac{1}{2} m \omega_0^{\,2} x^2 ,$$

where ω_0 is a constant. The normalized stationary states are given by $\varphi_0(x), \varphi_1(x), \varphi_2(x), \ldots$ for increasing eigenenergies $E_0, E_1, E_2, \ldots$, respectively.

At time $t = 0$ the system is described by the normalized wave function

$$\psi(x, t = 0) = \tfrac{1}{2}\sqrt{3}\, i\, \varphi_0(x) - K \varphi_2(x) ,$$

where K is a positive real-valued constant.

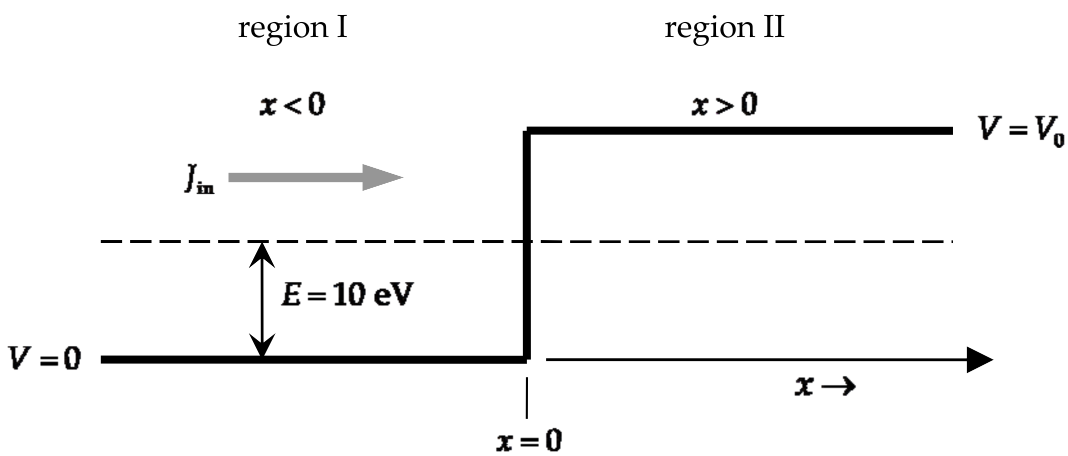
(a) Give the Hamiltonian for this system.

(b) Determine the value of K.

(c) If, at $t = 0$, we measure the energy, what is the probability we find the value E_0?

(d) Calculate the expectation value of the energy at $t = 0$.

Quantum Mechanics Group B (continued). Answer only 2 Group B questions.



B3. A beam of electrons (see above) with kinetic energy $E = 10$ eV impinges on a step with height $V_0 > E$. The incoming probability current is $J_{\text{in}} = 4.0 \times 10^5\ \text{s}^{-1}$ (left to right). We describe the wavefunction of the incoming electrons as $\varphi_{\text{in}}(x) = Ae^{ikx}$.

(a) Determine the quantities A and k, making sure to include units in your answers.

(b) Demonstrate that the wavefunction $\varphi_{\text{II}} = Be^{-\kappa x}$ is the solution of the time-independent Schrödinger equation in region II. Give an expression for κ .

(c) What is the probability current J_{II} in region II?

(d) What is the reflection coefficient R of this potential step?

At the position $x = d = 1.4$ Å (in region II) the absolute value of the *amplitude* of the evanescent wave has dropped to 1% of what it is at the location of the step ($x = 0$).

(e) What is the height V_0 of the step? Give your answer in eV.

Quantum Mechanics Group B (continued). Answer only 2 Group B questions.

B4. The wavefunction of a particle moving in one dimension (x) is given, at some time, by

$$\psi(x) = \begin{cases} K\, e^{ip_0 x/\hbar}\sqrt{b^2 - x^2} & \text{for } -b \le x \le b \\ 0 & \text{elsewhere} \end{cases}$$

in which p_0 and the normalization constant K are positive real numbers.

(a) Calculate the normalization constant K. What are the units of K (SI) ?

(b) For the given wavefunction, you measure the particle's position. Calculate the probability that you find it between $x = -\frac{1}{2}b$ and $x = 0$.

(c) Give a symmetry argument to explain why $\langle x \rangle = 0$ (no calculation).

(d) Calculate $\langle x^2 \rangle$.

(e) Calculate Δx.

Physical Constants

speed of light .... $c = 2.998 \times 10^{8}$ m/s	electrostatic constant ... $k = (4\pi\varepsilon_0)^{-1} = 8.988 \times 10^{9}$ m/F
Planck's constant .... $h = 6.626 \times 10^{-34}$ J·s	electron mass .... $m_{\mathrm{el}} = 9.109 \times 10^{-31}$ kg
Planck's constant / 2π ... $\hbar = 1.055 \times 10^{-34}$ J·s	proton mass .... $m_{\mathrm{p}} = 1.673 \times 10^{-27}$ kg
elementary charge .... $e = 1.602 \times 10^{-19}$ C	1 bohr............................. $a_0 = 0.5292$ Å
electric permittivity ...... $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m	1 hartree .... $E_{\mathrm{h}} = 27.21$ eV
magnetic permeability ... $\mu_0 = 1.257 \times 10^{-6}$ H/m	

EQUATIONS THAT MAY BE HELPFUL

Electrostatics:

$$\oiint_A \vec{E} \bullet d\vec{A} = \frac{q_{enc}}{\varepsilon_0} \qquad \vec{\nabla} \bullet \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{\nabla} X \vec{E} = 0 \qquad \vec{E} = -\nabla V$$

$$-\int_{r_1}^{r_2} \vec{E} \bullet d\vec{l} = V(r_2) - V(r_1) \qquad V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q(\vec{r}^{\,'})}{|\vec{r} - \vec{r}^{\,'}|}$$

Work done $W = -\int_{\vec{a}}^{\vec{b}} q\vec{E}.d\vec{l} = q[V(\vec{b}) - V(\vec{a})]$

Multipole expansion:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r}\int \rho(\vec{r}^{\,'})d\tau^{'} + \frac{1}{r^2}\int r^{'}Cos\theta^{'}\rho(\vec{r}^{\,'})d\tau^{'} + \frac{1}{r^3}\int (r^{'})^2(\frac{3}{2}Cos\theta^{'} - \frac{1}{2})\rho(\vec{r}^{\,'})d\tau^{'} + \ldots\right]$$

where the first term is the monopole term, the second is the dipole term, the third is the quadrupole term..; r and r' are the field point and source point as usual and θ' is the angle between them.

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P} \qquad \vec{\nabla} \bullet \vec{D} = \rho_f \qquad \rho_b = -\vec{\nabla} \bullet \vec{P} \qquad \sigma_b = \vec{P} \bullet \hat{n}$$

The above are true for ALL dielectrics. Confining ourselves to LIH dielectrics, we also have:

$$\vec{D} = \varepsilon\vec{E} \qquad \vec{P} = \chi_e\varepsilon_0\vec{E} \qquad \varepsilon = \varepsilon_0(1 + \chi_e) \qquad \kappa_e = \varepsilon/\varepsilon_0 \qquad \chi_e = \kappa_e - 1$$

C(dielectric)=κ_e C (vacuum)

Boundary Conditions:

$$E_{2t} - E_{1t} = 0 \qquad E_{2n} - E_{1n} = \frac{\sigma}{\varepsilon_0}$$

Magnetostatics:

Lorentz Force $\vec{F} = q(\vec{v} \times \vec{B}) + q\vec{E}$ Current densities $\vec{I} = \int \vec{J} \bullet d\vec{A}$ $\vec{I} = \int \vec{K} \bullet d\vec{l}$

Biot Savart Law $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l}' \times \hat{\Re}}{\Re^2}$ where R is the is the vector from the source point r' to the field point r.

This can also be written (for surface currents) as $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{K} \times \hat{\Re}}{\Re^2} \bullet dA'$

For a straight wire segment $B = \frac{\mu_0 I}{4\pi s}[Sin(\theta_2) - Sin(\theta_1)]$ where s is perpendicular distance from the wire.

For a circular loop of radius R, the B field at a point on the axis is given by

$$B = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z^2)^{3/2}}$$

For an infinitely long solenoid, the B field inside is given by $B=\mu_0 NI$ where N is the number of turns per unit length.

$\vec{\nabla} \bullet \vec{B} = 0$ $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ Ampere's law $\oint \vec{B} \bullet d\vec{l} = \mu_0 I_{enclosed}$

Magnetic vector potential A

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\nabla^2 \vec{A} = -\mu_0 \vec{J} \Rightarrow \vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}}{r - r'} d\tau'$$

Also for line and surface currents $\vec{A} = \frac{\mu_0}{4\pi} \int \frac{I}{r - r'} d\vec{l}'$ $\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{K}}{r - r'} da'$

From Stokes theorem $\oint \vec{A} \bullet d\vec{l} = \int \vec{B}.d\vec{a}$

For a magnetic dipole m , $\vec{A}_{dipole} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}$

Magnetic dipoles

Magnetic dipole moment of a current distribution is given by **m**=I∫d**a**

Torque on a magnetic dipole in a magnetic field **τ=mXB**

Force on a magnetic dipole $\vec{F} = \nabla(\vec{m} \bullet \vec{B})$

B field of a magnetic dipole $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{1}{r^3}[3(\vec{m} \bullet \hat{r})\hat{r} - \hat{m}]$

Dipole-dipole interaction energy is given by

$U_{DD} = \frac{\mu_0}{4\pi R^3} \left[(\vec{m}_1 \bullet \vec{m}_2) - 3(\vec{m}_1 \bullet \hat{R})(\vec{m}_2 \bullet \hat{R}) \right]$ where $\mathbf{R}=\mathbf{r}_1-\mathbf{r}_2$

<u>Material with magnetization M</u> produces a magnetic field equivalent to that of (bound)volume and surface current densities $\vec{J}_b = \vec{\nabla} X \vec{M}$ and $\vec{K}_b = \vec{M} \times \hat{n}$.

$\oint \vec{H} \bullet d\vec{l} = I_{freeenclosed}$ $\qquad \vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$

For linear magnetic material

M=χ_m**H** and **B**=$\mu_0(1+\chi_m)$**H** **or B**=μ**H**

<u>Boundary Conditions:</u> $B_{2n}-B_{1n}=0$ $\qquad B_{2//}-B_{1//}=\mu_0 K$

<u>Maxwell's Equations in vacuum:</u>

1.$\vec{\nabla} \bullet \vec{E} = \frac{\rho}{\varepsilon_0}$ Gauss' Law

2.$\vec{\nabla} \bullet \vec{B} = 0$

3.$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ Faradays law

4. $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ Ampere's Law with Maxwell's correction

<u>Maxwell's Equations in LIH media :</u>

1.$\vec{\nabla} \bullet \vec{E} = \frac{\rho_f}{\varepsilon}$ Gauss' Law

2.$\vec{\nabla} \bullet \vec{B} = 0$

3.$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ Faradays law

4. $\vec{\nabla} \times \vec{B} = \mu \vec{J}_f + \mu \varepsilon \frac{\partial \vec{E}}{\partial t}$ Ampere's Law with Maxwell's correction

Alternative ways of writing Faraday's Law $\oint \vec{E} \bullet d\vec{l} = -\frac{d\phi}{dt}$

Mutual and self inductance: $\phi_2=M_{21}I_1$ and $M_{21}=M_{12}$

ϕ=LI

Energy stored in a magnetic field: $W = \frac{1}{2\varepsilon_0} \int_V B^2 d\tau = \frac{1}{2} L I^2 = \frac{1}{2} \oint \vec{A} \bullet \vec{I} dl$

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

Curl : $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

- *Schrödinger Equation.* General and time-independent:

$$
\begin{aligned}
i\hbar\frac{\partial\Psi}{\partial t} &= \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\Psi \\
H\psi &= E\psi \\
\Psi &= \psi e^{-iEt/\hbar}
\end{aligned} \tag{1}
$$

- *Formalism and Operator Algebra.* This is probably actually the easiest topic for people who have taken 143a. Some equations of use are:

$$
\begin{aligned}
\langle\alpha|\beta\rangle &= \langle\beta|\alpha\rangle^* \\
\hat{O} &= \hat{O}^\dagger \text{ if } \hat{O} \text{ is Hermitian} \\
\langle\alpha|\hat{O}|\beta\rangle &= \beta\langle\alpha|\beta\rangle \text{ if } |\beta\rangle \text{ is an eigenstate of } \hat{O} \\
&= \int_{-\infty}^{\infty} \mathrm{d}x\, \alpha^*(x)\hat{O}(x)\beta(x) \text{ in 1D} \\
&= A^\dagger \times O \times B \text{ as matrices} \\
[\hat{A},\hat{B}] &= \hat{A}\hat{B} - \hat{B}\hat{A} \\
\therefore\ [\hat{A},\hat{B}] &= -[\hat{B},\hat{A}] \\
[\hat{A}\hat{B},\hat{C}] &= [\hat{A},\hat{C}]\hat{B} + \hat{A}[\hat{B},\hat{C}]
\end{aligned} \tag{2}
$$

A Hermitian matrix has real eigenvalues; observables thus must be represented by Hermitian operators.

- *Position and momentum.*

$$
\begin{aligned}
\hat{p} &= -i\hbar\nabla \\
[\hat{x},\hat{p}] &= i\hbar \\
[\hat{f}(x),\hat{p}] &= i\hbar\frac{\mathrm{d}f}{\mathrm{d}x}
\end{aligned} \tag{3}
$$

$$\psi(x) = \langle x|\psi\rangle = \int_{-\infty}^{\infty} dp\, \langle x|p\rangle\langle p|\psi\rangle = \int_{-\infty}^{\infty} dp\, \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}\,\varphi(p)$$

Mechanics Group A.
Solutions:

A1.

$$\boldsymbol{F} = \frac{1}{2}(x_o m\omega^2 sin\omega t)\hat{\boldsymbol{x}} + \left(mg + \frac{1}{2}y_o m\omega^2 cos\omega t\right)\hat{\boldsymbol{y}}$$

$$\boldsymbol{r} = (\mathrm{v}_o t + x_o sin\omega t)\hat{\boldsymbol{x}} + (\frac{1}{2}gt^2 + y_o cos\omega t)\hat{\boldsymbol{y}}$$

$$\dot{\boldsymbol{r}} = \frac{dr}{dt} = (\mathrm{v}_o + x_o\omega Cos(\omega t))\hat{\boldsymbol{x}} + (gt - y_o\omega Sin(\omega t))\hat{\boldsymbol{y}}\ ,$$

$$\ddot{\boldsymbol{r}} = \frac{d\dot{\boldsymbol{r}}}{dt} = (-x_o\omega^2 Sin(\omega t))\hat{\boldsymbol{x}} + (g - y_o\omega^2 Cos(\omega t))\hat{\boldsymbol{y}}$$

Since $m\ddot{\boldsymbol{r}} \neq \boldsymbol{F}$, then it is not inertial.

A2.

$$x = ACos(\omega t + \phi)$$

To stop the motion, we need to apply the pulse when the mass arrives at the equilibrium position, where all the potential energy converts onto kinetic energy. Then we have:

$$\frac{P^2}{2m} = \frac{1}{2}kA^2 = \frac{1}{2}m\omega^2 A^2$$

The impulse I is given by $I = P = m\omega A$.
To find the time, it is necessary to choose $x = 0$, then:

$$ACos(\omega t + \phi) = 0$$

$$\omega t + \phi = \frac{2n+1}{2}\pi, \qquad \text{with n an integer.}$$

Thus we get time as:

$$t = \frac{2n+1}{2\omega}\pi - \frac{\phi}{\omega}$$

A3. Suppose the total kinetic energy at the end of the ramp is E, and the radius of both cylinders is R, then:

$$E = \frac{1}{2}I_A\omega_A^2 + \frac{1}{2}mv_A^2 = \frac{1}{2}I_B\omega_B^2 + \frac{1}{2}mv_B^2$$

With $v_i = R\omega_i$. Then we have:

$$\frac{v_A^2}{v_B^2} = \frac{\omega_A^2}{\omega_B^2} = \frac{I_B + mR^2}{I_A + mR^2}$$

Then, if we know the relation between I_B and I_A, we can figure out the relation between its velocities.

A4. $$V(x) = \beta x^4 - \alpha x^2$$

a) The equilibria occur when $\frac{dV(x)}{dx} = 0$.

$$\frac{dV(x)}{dx} = 4\beta x^3 - 2\alpha x = 0$$

There exist three equilibrium points:

$x_1 = 0$, $x_2 = \sqrt{\frac{\alpha}{2\beta}}$ and $x_3 = -\sqrt{\frac{\alpha}{2\beta}}$

The stability is determined by $\frac{d^2V(x)}{dx^2}$, then:

$$\frac{d^2V(x)}{dx^2} = 12\beta x^2 - 2\alpha$$

To $x_1 = 0$:

$$\left.\frac{d^2V(x)}{dx^2}\right|_{x=x_1} = -2\alpha < 0$$

Then, $x_1 = 0$ is an unstable equilibrium point.

To $= \pm\sqrt{\frac{\alpha}{2\beta}}$:

$$\left.\frac{d^2V(x)}{dx^2}\right|_{x=x_{2,3}} = 4\alpha > 0$$

Then, the stable equilibrium occurs at $x = \pm\sqrt{\frac{\alpha}{2\beta}}$

b) The frequency of small oscillations is given by:

$$\omega = \sqrt{\frac{1}{m}\left(\left.\frac{d^2V(x)}{dx^2}\right|_{x=x_{2,3}}\right)}$$

$$\omega = 2\sqrt{\frac{\alpha}{m}}$$

c) Dimension for α and β are:

$$[\alpha] = Mass * (Time)^{-2}$$
$$[\beta] = Mass * (Length * Time)^{-2}$$

Mechanics Group B.
Solutions:

B1. Before of thrust:

$$m\frac{v^2}{r_1} = G\frac{Mm}{r_1^2}$$

$$L = mr_1 v$$

$$E = \frac{1}{2}mv^2 - G\frac{Mm}{r_1} = -\frac{1}{2}mv^2$$

$$\epsilon = \sqrt{1 + \frac{2EL^2}{m(GMm)^2}} = 0$$

After of thrust $v \to sv$

$$v_2 = sv$$

$$L_2 = mr_1 v_2$$

$$E_2 = \frac{1}{2}mv_2^2 - G\frac{Mm}{r_1} = \frac{s^2}{2}mv^2 - mv^2 = -\left(1 - \frac{s^2}{2}\right)mv^2$$

$$\epsilon_2 = \sqrt{1 + \frac{2E_2L_2^2}{m(GMm)^2}} = 1 - s^2$$

Then,

$$\frac{r_1}{r_2} = \frac{1+\epsilon_2}{1-\epsilon_2} = \frac{1+(1-s^2)}{1-(1-s^2)} = \frac{2-s^2}{s^2}$$

B2. $L = \frac{m}{2}[\dot{q}_1^2(1 + sin^2 q_2) + \dot{q}_2^2] - \frac{k}{3}q_2^3$

(a) Equations of motion are given by:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

Then:

$$q_1: \qquad \frac{d}{dt}[m\dot{q}_1(1 + sin^2 q_2)] = 0$$

$$q_2: \qquad \frac{d}{dt}[m\dot{q}_2] - M[\dot{q}_1^2(sin\, q_2 \cos q_2)] + kq_2^2 = 0$$

$$\Rightarrow \quad m\ddot{q}_2 - M[\dot{q}_1^2(sin\, q_2 \cos q_2)] + kq_2^2 = 0$$

(b) Yes, q_1 is cyclic. The conserved momentum is $m\dot{q}_1(1 + sin^2 q_2)$

(c) Hamiltonian is obtained by:

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} = m\dot{q}_1(1 + sin^2 q_2) \quad \Rightarrow \quad \dot{q}_1 = \frac{p_1}{m(1 + sin^2 q_2)}$$

$$p_2 = \frac{\partial L}{\partial \dot{q}_2} = m\dot{q}_2 \quad \Rightarrow \quad \dot{q}_2 = \frac{p_1}{m}$$

Then,

$$H = \sum_i p_i\, \dot{q}_i - L$$

$$H = \frac{p_1^2}{2m(1 + sin^2 q_2)} + \frac{p_2^2}{2m} + \frac{k}{3} q_2^3 = E$$

H is independent of time, thus energy is conserved.

B3. We have the relation between $\mu(x)$ and heat Q as $\mu(x) = \mu - kQ$, where k is the constant for this linear relation. Then $d\mu(x) = -kdQ$.

According to the conservation of total energy, the gravitational potential energy will be converted into kinetic energy and heat, which is

$$dG = dT + dQ$$

This can be interpreted as:

$$d\mu(mgx \sin\theta) = dT + d(\mu(x) mgx \cos\theta)$$

$$\Rightarrow \quad mg \sin\theta dx = dT + mgx \cos\theta\, d\mu(x) + \mu(x) mg \cos\theta\, dx$$

Also

$$dG = mg \sin\theta\, dx = dT - \frac{1}{k} d\mu(x)$$

Thus we get:

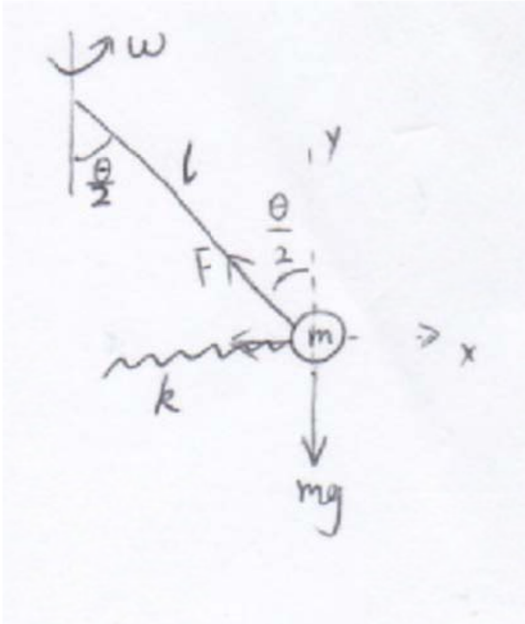
$$-\frac{1}{k} d\mu(x) = mgx \cos\theta\, d\mu(x) + \mu(x) mg \cos\theta\, dx$$

$$\Rightarrow \quad -\left(mgx \cos\theta + \frac{1}{k}\right) d\mu(x) = \mu(x) mg \cos\theta\, dx$$

$$\Rightarrow \quad \int_{\mu}^{\mu(s)} \frac{d\mu(x)}{\mu(x)} = -\int_0^s \frac{kmg \cos\theta}{kmgx \cos\theta + 1} dx$$

$$\mu(s) = \frac{\mu}{kmgs \cos\theta + 1}$$

B4.



According to the force balance, we get:

$$x: \qquad m\left(L\sin\frac{\theta}{2}\right)\omega^2 = F\sin\frac{\theta}{2} + kL\sin\frac{\theta}{2}$$

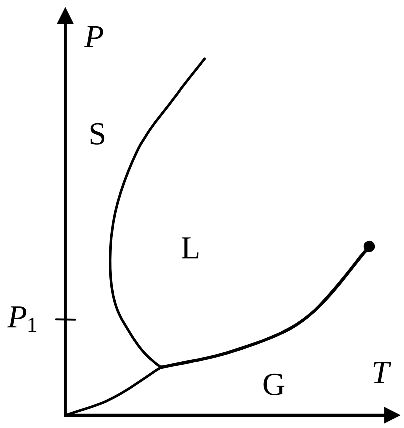
$$y: \qquad mg = F\cos\frac{\theta}{2}$$

Then, we get

$$\cos\frac{\theta}{2} = \frac{mg}{(m\omega^2 - k)L} \quad \Rightarrow \quad \theta = 2\arccos\left[\frac{mg}{(m\omega^2 - k)L}\right]$$

Thermodynamics and Statistics Group A.
Solutions:

A2.



At P_1 the solid float.

A3. $P = P(V,T)$

$$dP = \left(\frac{\partial P}{\partial V}\right)_T dV + \left(\frac{\partial P}{\partial T}\right)_V dT$$

Now, using the relations:

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$$

And

$$\left(\frac{\partial x}{\partial y}\right)_z = \frac{1}{\left(\frac{\partial y}{\partial x}\right)_z}$$

$$\Rightarrow \quad \left(\frac{\partial P}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_P \left(\frac{\partial T}{\partial P}\right)_V = -1$$

$$\Rightarrow \quad \left(\frac{\partial P}{\partial T}\right)_V = -\left(\frac{\partial P}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_P$$

A4.

A) The Kelvin statement of the second law.

“It is impossible to construct an engine that, operating in a cycle, will produce no effect other than the extraction of heat from a reservoir and the performance of an equivalent amount of work”. (Zemanzky-Thermodynamics)

B) The Clausius statement of the second law.

“It is impossible to construct a refrigerator that, operating in a cycle, will produce no effect other than transfer of heat from a lower-temperature reservoir to a higher-temperature reservoir”. (Zemanzky-Thermodynamics)

C) The entropy statement of the second law.

For any kind of process the entropy of the Universe follows $\Delta S \geq O$. Always the entropy increase or at least keep constant, but never decrease (Universe is the whole system). In some parts of the system the entropy can decrease, but for the whole system increase or keep constant.

$\Delta S = O$ reversible process

$\Delta S > O$ irreversible process

(Zemanzky-Thermodynamics)

Thermodynamics and Statistics Group B.
Solutions:

B1.

a) At the equilibrium, the two blocks reach the same temperature T_f. For the heat engine we have:

$$\eta = \frac{W}{Q_h} = \frac{Q_h - Q_c}{Q_h} = \frac{T_h - T_c}{T_h} \Rightarrow \frac{Q_h}{Q_c} = \frac{T_h}{T_c}$$

Also

$$Q_c = C(T_f - T_c), \quad Q_h = C(T_h - T_f)$$

Then we get:

$$T_f = \frac{2T_hT_c}{T_h + T_c}$$

b) $\Delta S = \Delta S_h + \Delta S_c = 0$

B2.

$$Q_m = aQ_h \Rightarrow T_m dS = aT_h dS_h \Rightarrow \frac{dS_h}{dS_m} = \frac{T_m}{aT_h}$$

$$\eta = \frac{W}{Q} = \frac{-(Q_h + Q_m) - Q_c}{-(Q_h + Q_m)} = 1 - \frac{Q_c}{-(Q_h + Q_m)} = 1 - \frac{T_c dS_c}{-(T_h dS_h + T_m dS_m)}$$

Since $dS_c = -(dS_h + dS_m)$, then

$$\eta = 1 - \frac{T_c(dS_h + dS_m)}{T_h dS_h + T_m dS_m} = 1 - \frac{T_c\left(\frac{dS_h}{dS_m} + 1\right)}{T_h \frac{dS_h}{dS_m} + T_m}$$

$$\eta = 1 - \frac{T_cT_m + aT_cT_h}{T_hT_m + aT_hT_m} = 1 - \frac{T_c(T_m + aT_h)}{T_hT_m(1 + a)}$$

For limits $a \to 0$ and $T_m \to T_c$, we have:

$$\eta = 1 - \frac{T_c}{T_h}$$

B3.

(a) No

Cooling of water = melted amount of ice

Then,

$$4186\ ^{J}/_{(kgK)} \text{ x } 1\ kg \text{ x } 20\ K = m_{ice} \text{ x } 334\ ^{J}/_{g} \Rightarrow m_{ice} = 250.66\ g$$

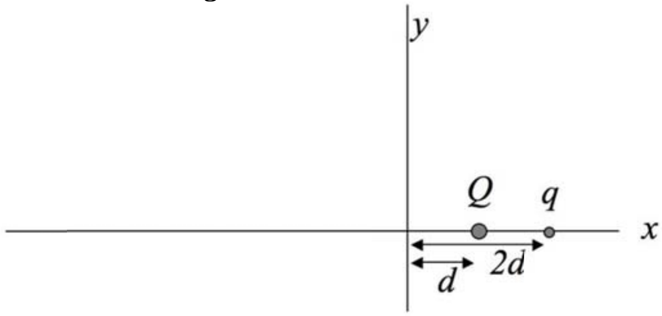
(b)

$$\Delta S = \Delta S_w + \Delta S_m = N_w C_w \ln\frac{T}{T_o} + \frac{Q}{T_{ice}}$$

$$\Delta S = 1\ kg \text{ x } 4186\ ^{J}/_{(kgK)} \text{ x } \ln\frac{273}{293} + \frac{4186 \text{ x } 1 \text{ x } 20}{273}\ ^{J}/_{K} = 10.713\ ^{J}/_{K}$$

<u>Electricity and Magnetism Group A.</u>
<u>Solutions:</u>

A1. Case 1: charge Q at $x = d$



$$\boldsymbol{F_{Qq}} = \frac{1}{4\pi\varepsilon_o}\frac{-2q^2}{d^2}\hat{\boldsymbol{x}} = -\frac{1}{4\pi\varepsilon_o}2\left(\frac{q}{d}\right)^2\hat{\boldsymbol{x}}$$

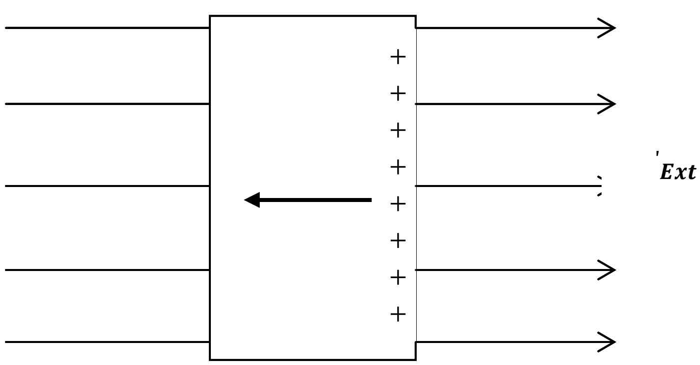
Case 2: An infinite, grounded plane conductor is placed in the *xy*-plane

$$\boldsymbol{F_{Plane-q}} = \frac{1}{4\pi\varepsilon_o}\frac{-q^2}{(4d)^2}\hat{\boldsymbol{x}} = -\frac{1}{4\pi\varepsilon_o}\frac{1}{16}\left(\frac{q}{d}\right)^2\hat{\boldsymbol{x}}$$

$\boldsymbol{F_{Qq}} > \boldsymbol{F_{Plane-q}}$, both have the same direction.

$$\left|\frac{\boldsymbol{F_{Qq}}}{\boldsymbol{F_{Plane-q}}}\right| = \frac{\frac{1}{4\pi\varepsilon_o}2\left(\frac{q}{d}\right)^2}{\frac{1}{4\pi\varepsilon_o}\frac{1}{16}\left(\frac{q}{d}\right)^2} = 32$$

A2.



$$\boldsymbol{E_T} = \boldsymbol{E_{Ext}} + \boldsymbol{E_{int}}$$

Then, the electric field is decreased.

Now, $U_E \propto |\boldsymbol{E}|^2 \;\Rightarrow\; U_E$ decreases.

A3.
We have:

$$\boldsymbol{E} = E_{\boldsymbol{o}}\hat{\boldsymbol{x}}$$

According with the Lorentz force:

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

Then:

$$\boldsymbol{B} = B_{\boldsymbol{o}}(\pm\hat{\boldsymbol{x}})$$

And the charge is positive.

A4.

$$\boldsymbol{m} = \frac{1}{2}\int_{\Omega} \mathbf{r} \times \mathbf{K}\, da$$

$\mathbf{r} = r\,\hat{\boldsymbol{r}}$ and $\mathbf{K} = \mathrm{K}_{\mathbf{o}}\,\hat{\boldsymbol{\varphi}}$

Then:

$$\boldsymbol{m} = \frac{1}{2}\int_{a}^{b} r\,\hat{\boldsymbol{r}} \times \mathrm{K}_{\mathbf{o}}\,\hat{\boldsymbol{\varphi}}\,(2\pi r dr)$$

$$\boldsymbol{m} = \pi \mathrm{k}_{\mathbf{o}} \int_{a}^{b} r^2 dr\,\hat{\boldsymbol{z}}$$

$$\boldsymbol{m} = \frac{\pi \mathrm{k}_{\mathbf{o}}}{3}(b^3 - a^3)\hat{\boldsymbol{z}}$$

Electricity and Magnetism Group B.
Solutions:

B1.

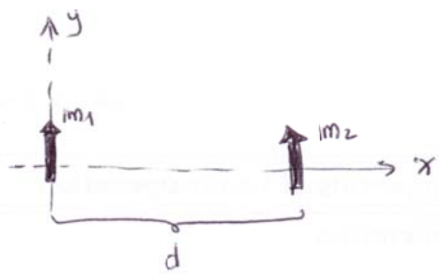
(a) Cylindrical symmetry requires B–field in the direction of $\hat{\boldsymbol{\varphi}}$ and has the same magnitude for the same distance from the z–axis.
(b) The Ampere loop is coaxial with the cylinder.
Inside ($r \leq R$):

$$\oint \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 I$$
$$\Rightarrow B 2\pi r = \mu_0 J_0 \pi r^2$$
$$\Rightarrow \boldsymbol{B}(r \leq R) = \frac{\mu_0 J_0 r}{2} \hat{\boldsymbol{\varphi}}$$

Outside ($r > R$):

$$\Rightarrow B 2\pi r = \mu_0 J_0 \pi R^2$$
$$\Rightarrow \boldsymbol{B}(r > R) = \frac{\mu_0 J_0 R^2}{2r} \hat{\boldsymbol{\varphi}}$$

B2. (a)



$$\boldsymbol{m_1} = m_1 \hat{\boldsymbol{y}}$$
$$\boldsymbol{m_2} = m_2 \hat{\boldsymbol{y}}$$

(b)

$$\boldsymbol{\tau} = \boldsymbol{m_2} \times \boldsymbol{B_1}$$
$$\boldsymbol{B_1} = \frac{\mu_0}{4\pi} \frac{1}{x^3} [3(\boldsymbol{m_1} \cdot \hat{\boldsymbol{x}})\hat{\boldsymbol{x}} - \boldsymbol{m_1}]$$
$$\boldsymbol{B_1} = -\frac{\mu_0}{4\pi x^3} \boldsymbol{m_1}$$
$$\Rightarrow \boldsymbol{\tau} = \boldsymbol{m_2} \times \left[-\frac{\mu_0}{4\pi x^3} \boldsymbol{m_1}\right] = -\frac{\mu_0}{4\pi x^3} \boldsymbol{m_2} \times \boldsymbol{m_1} = 0$$

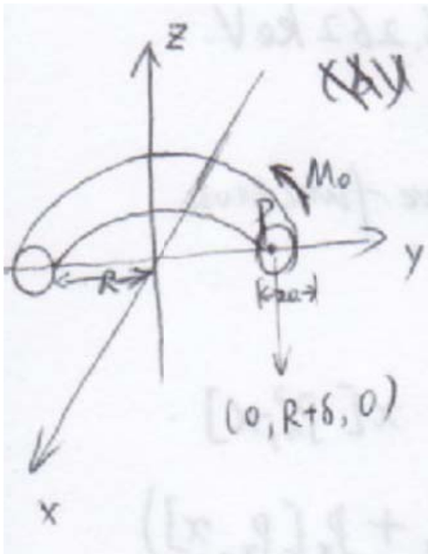
No rotation of $\boldsymbol{m_2}$ due to $\boldsymbol{m_1}$
(c) $\boldsymbol{F} = \boldsymbol{\nabla}(\boldsymbol{m_2} \cdot \boldsymbol{B})|_{x=d}$

$$\boldsymbol{m_2}\cdot\boldsymbol{B}=\boldsymbol{m_2}\cdot\left(-\frac{\mu_0}{4\pi x^3}\boldsymbol{m_1}\right)==-\frac{\mu_0 m_1 m_2}{4\pi x^3}$$
$$\boldsymbol{\nabla}(\boldsymbol{m_2}\cdot\boldsymbol{B})=\frac{3\mu_0 m_1 m_2}{4\pi x^4}\hat{\boldsymbol{x}}$$
$$\Rightarrow \boldsymbol{F}=\frac{3\mu_0 m_1 m_2}{4\pi d^4}\hat{\boldsymbol{x}}$$

$\boldsymbol{m_2}$ is repelled by $\boldsymbol{m_1}$

B3.

(a)

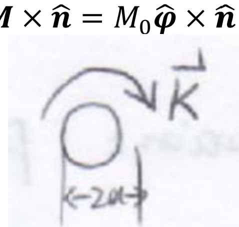


$$\boldsymbol{J}_b=\boldsymbol{\nabla}\times\boldsymbol{M}=0$$

No bound volume current, since $\boldsymbol{M}=M_0\hat{\boldsymbol{\varphi}}$

(b)

$$\boldsymbol{K}_b=\boldsymbol{M}\times\hat{\boldsymbol{n}}=M_0\hat{\boldsymbol{\varphi}}\times\hat{\boldsymbol{n}}=M_0\hat{\boldsymbol{K}}$$

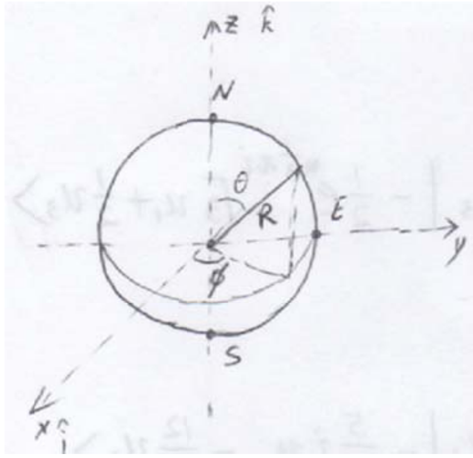


(c) $B_{2\,||}-B_{1\,||}=\mu_0 K$

$\Rightarrow B=\mu_0 M_0$

$\boldsymbol{B}=\mu_0 M_0\hat{\boldsymbol{\varphi}}$ along axis of cylinder

B4. (a)



$$N: \quad \theta = 0 \quad \Rightarrow \quad E_r = \frac{2A}{r^3}, \; E_\theta = 0, \; E_\phi = 0 \quad \uparrow \hat{\boldsymbol{k}}$$

$$S: \quad \theta = \pi \quad \Rightarrow \quad E_r = -\frac{2A}{r^3}, \; E_\theta = 0, \; E_\phi = 0 \quad \uparrow \hat{\boldsymbol{k}}$$

$$E: \quad \theta = \frac{\pi}{2} \quad \Rightarrow \quad E_r = 0, \; E_\theta = \frac{A}{r^3}, \; E_\phi = 0 \quad \downarrow \hat{\boldsymbol{k}}$$

(b)

$$\rho = \varepsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E}$$

$$\rho = \varepsilon_0 \left[\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 E_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta \, E_\theta) + \frac{1}{r \sin\theta} \frac{\partial E_\phi}{\partial \phi} \right]$$

$$\rho = \varepsilon_0 \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(\frac{2A \cos\theta}{r} \right) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} \left(\frac{2A \sin^2\theta}{r^3} \right) \right]$$

$$\rho = 0$$

(c) On the surface of the sphere, surface charge density σ, no volume charge density

$E_{2n} - E_{1n} = \frac{\sigma}{\varepsilon_0} \Rightarrow \sigma = \varepsilon_0 E_r = \frac{2\varepsilon_0 A \cos\theta}{r^3}$, antisotropic

Quantum Mechanics Group A.
Solutions:

A1. For photon $E_p = pc$, then for the electron $E_e = \frac{p^2}{2m_e} = \frac{E_p^2}{2m_e c^2}$

$$E_e = \frac{(80\text{x}10^3\text{x}1.602\text{x}10^{-19})^2}{2\text{x}9.109\text{x}10^{-31}\text{x}(2.998\text{x}10^8)^2} = 1.003\text{x}10^{-15}\text{ J} = 6.262\ keV$$

A2. $P(left) = \frac{1}{2}$, because of the symmetry of the wave function.

A3.

$$[x^2, p_x^2] = [x, p_x^2]x + x[x, p_x^2] = -[p_x^2, x]x - x[p_x^2, x]$$
$$[x^2, p_x^2] = -([p_x, x]p_x + p_x[p_x, x])x - x([p_x, x]p_x + p_x[p_x, x])$$
$$[x^2, p_x^2] = 2i\hbar p_x x + 2i\hbar x p_x = 2i\hbar$$

Thus, it is not Hermitian

A4. (a)

$$|\psi\rangle = -\frac{1}{2}e^{\frac{1}{3}\pi i}\sqrt{3}\ |u_1\rangle + \frac{1}{2}\ |u_2\rangle$$
$$\langle\psi|\psi\rangle = \left(-\frac{1}{2}e^{-\frac{1}{3}\pi i}\sqrt{3}\langle u_1| + \frac{1}{2}\langle u_1|\right)\left(-\frac{1}{2}e^{\frac{1}{3}\pi i}\sqrt{3}\ |u_1\rangle + \frac{1}{2}\ |u_2\rangle\right)$$
$$\langle\psi|\psi\rangle = \frac{3}{4} + \frac{1}{4} = 1$$

⇨ $|\psi\rangle$ is normalized.

(b)

$$|s\rangle = -\frac{5}{13}i\ |u_2\rangle + \frac{12}{13}\ |u_3\rangle$$
$$P(s) = |\langle s|\psi\rangle|^2$$
$$P(s) = \left|\left(-\frac{5}{13}i\langle u_2| + \frac{12}{13}\langle u_3|\right)\left(-\frac{1}{2}e^{\frac{1}{3}\pi i}\sqrt{3}\ |u_1\rangle + \frac{1}{2}\ |u_2\rangle\right)\right|^2$$
$$P(s) = \left|-\frac{12}{26}\right|^2 = \frac{36}{169}$$

Quantum Mechanics Group B.
Solutions:

B1. (a) According to normalization $\int_{-\infty}^{+\infty} \varphi^*(p)\varphi(p)dp = 1$, we get:

$$\int_{-\frac{h}{2a}}^{+\frac{h}{2a}} K^2 dp = 1 \quad \Rightarrow \quad K = \sqrt{\frac{a}{h}}$$

(b)

$$\psi(x) = \langle x|\psi\rangle = \int_{-\infty}^{+\infty} dp\langle x|p\rangle\langle p|x\rangle = \int_{-\frac{h}{2a}}^{+\frac{h}{2a}} dp \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}} K$$

$$\psi(x) = \frac{K}{ix}\sqrt{\frac{2\hbar}{\pi}} \sin\frac{\pi x}{a} = \frac{\sqrt{a}}{i\pi x}\sin\frac{\pi x}{a}$$

B2. (a)

$$H = T + V = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega_0^2 x^2$$

(b) Normalization:

$$\langle\psi|\psi\rangle = 1 = \langle -\frac{1}{2}\sqrt{3}i\varphi_0 - K\varphi_2 \left|\frac{1}{2}\sqrt{3}i\varphi_0 - K\varphi_2\right\rangle = \frac{3}{4} + K^2$$

Then, $K = \frac{1}{2}$

(c) Since $E_0 \to \varphi_0$, then $P(t = 0, E_0) = \left|\frac{1}{2}\sqrt{3}i\right|^2 = \frac{3}{4}$

(d) $\langle E\rangle = P(E_0)E_0 + P(E_2)E_2 = \frac{3}{4}E_0 + \frac{1}{4}E_2 = \frac{3}{4}\left(\frac{1}{2}\hbar\omega_0\right) + \frac{1}{4}\left(\frac{5}{2}\hbar\omega_0\right) = \hbar\omega_0$

B3. (a)

$$k = \sqrt{\frac{2m_e E}{\hbar^2}} = \frac{\sqrt{2\text{x}9.109\text{x}10^{-31}\text{x}10\text{x}1.602\text{x}10^{-19}\,J}}{1.055\text{x}10^{-34}\,J\,s} = 1.6193\text{x}10^{10}\ \text{m}^{-1}$$

$$J_{in} = -\frac{i\hbar}{2m}\left(\psi_{in}^*\frac{d}{dx}\psi_{in} - \psi_{in}\frac{d}{dx}\psi_{in}^*\right) = \frac{k\hbar A^2}{m} \quad \Rightarrow \quad A = \sqrt{\frac{mJ_{in}}{k\hbar}}$$

$$A = \sqrt{\frac{9.109\text{x}10^{-31}\,kg\ \text{x}\ 4.0\text{x}10^5\,s^{-1}}{1.6193\text{x}10^{10}\ \text{m}^{-1}\text{x}1.055\text{x}10^{-34}\,J\,s}} = 0.462\,m^{-\frac{1}{2}}$$

(b)

$H_{II} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_0$ and $H_{II}\psi_{II} = E\psi_{II}$. Then we get:

$$\frac{d^2\psi_{II}}{dx^2} = \frac{2m(V_0 - E)}{\hbar^2}\psi_{II}$$

$$\Rightarrow \psi_{II} = Be^{-Kx}, \qquad K = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

(c) The boundary conditions are at $x = 0$

$$\psi_I = \psi_{II}, \quad \psi'_{II} = \psi'_I$$

$$\Rightarrow A + B = C, \quad ikA - ikC = -KB$$

$$B = \frac{2kA}{k + iK}, \quad \Rightarrow J_{II} = 0$$

(d)

$$C = \frac{k - iK}{k + iK} A, \quad \Rightarrow R = \frac{|C|^2}{|A|^2} = 1$$

(e)

$$e^{-Kd} = 1\%, \quad \Rightarrow V_0 = E + \frac{2\hbar^2}{md^2}(m\,10)^2 = 51.265\ eV$$

B4. (a) Normalization $\langle \psi | \psi \rangle = 1$

$$\Rightarrow \int_{-b}^{b} K^2 (b^2 - x^2) dx = 1 \quad \Rightarrow K = \frac{1}{2b}\sqrt{\frac{3}{b}} \qquad (m^{-\frac{3}{2}})$$

(b)

$$P = \int_{-\frac{b}{2}}^{0} \psi^* \psi\, dx = K^2\, b^2 x - \frac{x^3}{3}\Big|_{-\frac{b}{2}}^{0} = \frac{11}{32}$$

(c) $(b^2 - x^2)$ is an even function which is symmetrical about $x = 0$, then $x(b^2 - x^2)$ is an odd function which is antisymmetrical.

(d) $\langle x^2 \rangle = \int_{-b}^{b} K^2 x^2 (b^2 - x^2) dx = \frac{b^2}{5}$

(e) $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{b}{\sqrt{5}}$