UNL Department of Physics and Astronomy
Preliminary Examination – Day 2
10 August 2012

This test covers the topics of Electricity and Magnetism (Topic 1) and Quantum Mechanics (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work 2 problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

E&M Group A. Answer only 2 Group A questions.

A1. A cylinder of radius R and length L, centered at the origin with the symmetry axis along the z axis, carries a polarization in the x-direction given by $\boldsymbol{P} = (ax + by)\hat{\boldsymbol{x}}$. Calculate the volume and surface bound charges as functions of the azimuthal angle φ.

A2. 100 parallel plate capacitors of capacitance 0.5 F each are connected in series across a 20V d.c. power supply.

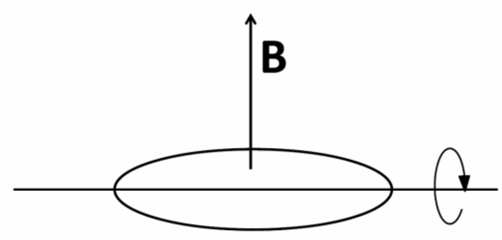
(a) Find the energy stored in one capacitor and also in the system of capacitors.

(b) Give the answer for the case of parallel connection.

A3. A closed circular loop of wire of radius 10 cm and electrical resistance 2 Ω is placed in a uniform magnetic field of 10^{-3}T which is initially perpendicular to the loop's plane. The loop turns around a diameter.

(a) What is the net charge that has passed through the loop during half a revolution?

(b) During one complete revolution?



A4. Two parallel infinite planar conductors carry uniform parallel surface currents, each with magnitude K. Both conductors are parallel to the xy plane and the surface currents are both flowing in the $\hat{x}$ direction. Find the magnetic field in the whole space, that is above the upper plane, between the planes and below the lower plane. Indicate both magnitude and direction.

E&M Group B. Answer only 2 Group B questions.

B1. Two concentric isolated spherical shells of radii r_1 and r_2 have charges Q_1 and Q_2. Find the electric field and potential for

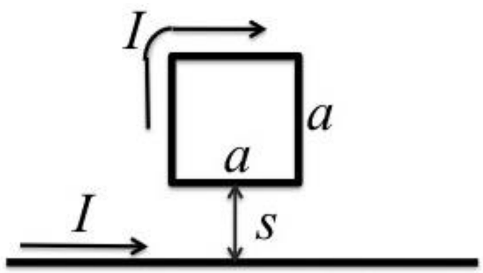
a) $r < r_1$; b) $r_1 < r < r_2$; c) $r > r_2$.

Analyze your answer when $Q_2 = -Q_1$.

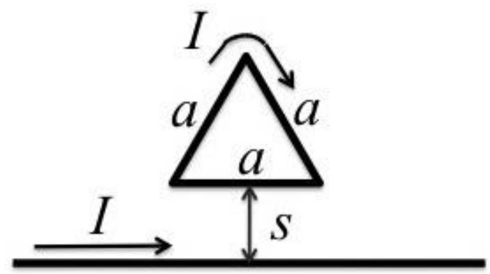
B2.

(a) Find the total force on the square loop placed as shown below in Fig. (a) near an infinite straight wire. Both the loop and wire carry a steady current I.

(b) Find the force on the triangular loop pictured below in Fig. (b) near an infinite straight wire.



(a)



(b)

B3. Consider an infinitely long cylinder of radius R. It carries a ***free*** current I uniformly distributed over the cross section of the cylinder and a magnetization **M** given by $\mathbf{M} = ks\hat{\boldsymbol{\varphi}}$, where s is the radial distance from the axis. Find the **H** and **B vectors** inside the cylinder.

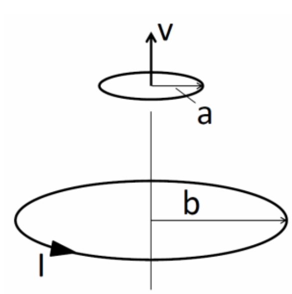
B4. A small loop of wire of radius a moves away from a large loop of radius b along the line joining the loops' centers. Each loop remains perpendicular to the central line and the distance between the loops varies as $z=z_0+vt$. A constant current I is maintained in the large loop.

(a) Find the current in the small loop I' as a function of time t if its electrical resistance is R. Neglect the effect of self-inductance.

(b) Show the direction of the current in the small loop.

(c) Find the behavior of the current at large time, that is at $vt >> z_0$ and $vt >> b$.

(d) Compute the ratio I'/I if a =1 cm, b = 50 cm, z_0 = 0, v = 1 m/s, t = 4s, and R = 0.2 Ω. Show your work.



Quantum Mechanics Group A. Answer only 2 Group A questions.

A1. A quantum oscillator with mass m and frequency ω is in its first excited state. Find the range of coordinate x allowed for a classical oscillator with the same energy.

A2. A particle of mass m is bound in a finite square well of height V and width L. Assuming that $\hbar^2/mL^2 << V$, estimate the mass's ground-state energy to the zeroth order of the parameter $\hbar^2/(mL^2V)$.

A3. The 3p→2s radiative transition in the hydrogen atom has wavelength 656 nm. Find the wavelength of the same transition in He^+ and positronium (the e^-e^+ system).

A4. The nucleus of the bromine atom has diameter $D = 1.0 \times 10^{-14}$ m. Use this information to estimate the minimum kinetic energy of a neutron in this nucleus (in eV).

Quantum Mechanics Group B. Answer only 2 Group B questions.

B1. Consider a one-dimensional harmonic oscillator (point particle in a parabolic potential) with normalized energy eigenfunctions $\varphi_n(x)$ and corresponding eigenenergies $(n+\frac{1}{2})\hbar\omega_0$ ($n = 0, 1, 2, \ldots$). At time $t = 0$, its wavefunction is given by

$$\psi(x,t=0) = \tfrac{1}{2}\,\varphi_0(x) + \tfrac{1}{2}\,\varphi_2(x) + \tfrac{1}{2}i\,\varphi_4(x) - \tfrac{1}{2}i\,\varphi_5(x)$$

a. Show that this wavefunction is normalized.

b. Calculate the expectation value of the energy for $t > 0$.

The parity operator $\hat{P}$ is defined as $\hat{P}\varphi(x) = \varphi(-x)$.

c. Demonstrate that $\hat{P}$ is Hermitian.

d. Demonstrate that the *only* eigenvalues (η) of $\hat{P}$ are $\eta = -1$ and $\eta = +1$.
One way to do this is by considering $\hat{P}\hat{P}$.

e. Calculate the expectation value of $\hat{P}$.

B2. A quantum system is in simultaneous eigenstates of $\mathbf{L}^2$ and L_z with eigenvalues $l(l+1)\hbar^2$ and $m\hbar$, respectively.

(a) What is the expectation value of L^2_x? Of L_+? Of L_x?

(b) Find the uncertainty ΔL_x for the maximum possible value of m and explain your result in terms of the commutation relations.

Reminder: $L_+ = L_x + iL_y$.

B3. A hydrogen atom is in the 2p state. Find

(a) the degeneracy of this state;

(b) all possible values of the quantum number j and the total angular momentum **J**;

(c) all possible values of m_j;

(d) the most probable distance of the electron from the nucleus;

(e) the average distance of the electron from the nucleus;

(f) calculate the average distance for the 2s state and compare your result with that obtained in (e).

B4. An electron source emits electrons of kinetic energy $E = 12$ eV. This electron beam impinges on a rectangular step-up potential where the potential energy increases from 0 to V, with V >12 eV. At a distance $d = 1.7$ Å beyond the potential step the amplitude of the electron wave function has dropped to 4.0% of what it is at the location of the step.

Calculate the step height V of the step-up potential in eV.

Physical Constants

speed of light .... $c = 2.998 \times 10^8$ m/s
electrostatic constant $k = (4\pi\varepsilon_0)^{-1} = 8.988 \times 10^9$ m/F
Planck's constant .... $h = 6.626 \times 10^{-34}$ J·s
electron mass .... $m_{el} = 9.109 \times 10^{-31}$ kg
Planck's constant / 2π ... $\hbar = 1.055 \times 10^{-34}$ J·s
proton mass .... $m_p = 1.673 \times 10^{-27}$ kg
elementary charge .... $e = 1.602 \times 10^{-19}$ C
1 bohr .... $a_0 = 0.5292$ Å
electric permittivity ...... $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m
1 hartree .... $E_h = 27.21$ eV
magnetic permeability ... $\mu_0 = 1.257 \times 10^{-6}$ H/m

neutron mass = 1.7×10^{-27} kg

EQUATIONS THAT MAY BE HELPFUL

Electrostatics:

$$\oint\!\!\oint_A \vec{E} \bullet d\vec{A} = \frac{q_{enc}}{\varepsilon_0} \qquad \vec{\nabla} \bullet \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{\nabla} X \vec{E} = 0 \qquad \vec{E} = -\nabla V$$

$$-\int_{r_1}^{r_2} \vec{E} \bullet d\vec{l} = V(r_2) - V(r_1) \qquad V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}$$

Work done $W = -\int_{\vec{a}}^{\vec{b}} q\vec{E}.d\vec{l} = q[V(\vec{b}) - V(\vec{a})]$

Multipole expansion:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r}\int \rho(\vec{r}\,')d\tau' + \frac{1}{r^2}\int r' Cos\theta' \rho(\vec{r}\,')d\tau' + \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}Cos\theta' - \frac{1}{2}\right)\rho(\vec{r}\,')d\tau' + \ldots\right]$$

where the first term is the monopole term, the second is the dipole term, the third is the quadrupole term..; r and r' are the field point and source point as usual and θ' is the angle between them.

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} \qquad \vec{\nabla} \bullet \vec{D} = \rho_f \qquad \rho_b = -\vec{\nabla} \bullet \vec{P} \qquad \sigma_b = \vec{P} \bullet \hat{n}$$

The above are true for ALL dielectrics. Confining ourselves to LIH dielectrics, we also have:

$$\vec{D} = \varepsilon \vec{E} \qquad \vec{P} = \chi_e \varepsilon_0 \vec{E} \qquad \varepsilon = \varepsilon_0(1 + \chi_e) \qquad \kappa_e = \varepsilon/\varepsilon_0 \qquad \chi_e = \kappa_e - 1$$

C(dielectric)=κ_e C (vacuum)
Boundary Conditions:

$$E_{2t} - E_{1t} = 0 \qquad E_{2n} - E_{1n} = \frac{\sigma}{\varepsilon_0}$$

Magnetostatics:

Lorentz Force $\vec{F} = q(\vec{v} \times \vec{B}) + q\vec{E}$ Current densities $\vec{I} = \int \vec{J} \bullet d\vec{A}$ $\vec{I} = \int \vec{K} \bullet d\vec{l}$

Biot Savart Law $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l}' \times \hat{\Re}}{\Re^2}$ where R is the is the vector from the source point r' to the field point r.

This can also be written (for surface currents) as $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{K} \times \hat{\Re}}{\Re^2} \bullet dA'$

For a straight wire segment $B = \frac{\mu_0 I}{4\pi s}[Sin(\theta_2) - Sin(\theta_1)]$ where s is perpendicular distance from the wire.

For a circular loop of radius R, the B field at a point on the axis is given by

$$B = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z^2)^{3/2}}$$

For an infinitely long solenoid, the B field inside is given by B=μ_0NI where N is the number of turns per unit length.

$\vec{\nabla} \bullet \vec{B} = 0$ $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ Ampere's law $\oint \vec{B} \bullet d\vec{l} = \mu_0 I_{enclosed}$

Magnetic vector potential A

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\nabla^2 \vec{A} = -\mu_0 \vec{J} \Rightarrow \vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}}{r - r'} d\tau'$$

Also for line and surface currents $\vec{A} = \frac{\mu_0}{4\pi} \int \frac{I}{r - r'} d\vec{l}'$ $\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{K}}{r - r'} da'$

From Stokes theorem $\oint \vec{A} \bullet d\vec{l} = \int \vec{B}.d\vec{a}$

For a magnetic dipole m , $\vec{A}_{dipole} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}$

Magnetic dipoles

Magnetic dipole moment of a current distribution is given by **m**=I∫d**a**

Torque on a magnetic dipole in a magnetic field **τ=mXB**

Force on a magnetic dipole $\vec{F} = \nabla(\vec{m} \bullet \vec{B})$

B field of a magnetic dipole $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{1}{r^3}[3(\vec{m} \bullet \hat{r})\hat{r} - \hat{m}]$

Dipole-dipole interaction energy is given by

$U_{DD} = \frac{\mu_0}{4\pi R^3}\left[(\vec{m}_1 \bullet \vec{m}_2) - 3(\vec{m}_1 \bullet \hat{R})(\vec{m}_2 \bullet \hat{R})\right]$ where $\mathbf{R}=\mathbf{r}_1-\mathbf{r}_2$

Material with magnetization M produces a magnetic field equivalent to that of (bound)volume and surface current densities $\vec{J}_b = \vec{\nabla} X \vec{M}$ and $\vec{K}_b = \vec{M} \times \hat{n}$.

$\oint \vec{H} \bullet d\vec{l} = I_{freeenclosed}$ $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$

For linear magnetic material

M=χ_m**H** and **B**=$\mu_0(1+\chi_m)$**H** **or B**=μ**H**

Boundary Conditions: $B_{2n}-B_{1n}=0$ $B_{2//}-B_{1//}=\mu_0 K$

Maxwell's Equations in vacuum:

1. $\vec{\nabla} \bullet \vec{E} = \frac{\rho}{\varepsilon_0}$ Gauss' Law
2. $\vec{\nabla} \bullet \vec{B} = 0$
3. $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ Faradays law
4. $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ Ampere's Law with Maxwell's correction

Maxwell's Equations in LIH media :

1. $\vec{\nabla} \bullet \vec{E} = \frac{\rho_f}{\varepsilon}$ Gauss' Law
2. $\vec{\nabla} \bullet \vec{B} = 0$
3. $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ Faradays law
4. $\vec{\nabla} \times \vec{B} = \mu \vec{J}_f + \mu\varepsilon \frac{\partial \vec{E}}{\partial t}$ Ampere's Law with Maxwell's correction

Alternative ways of writing Faraday's Law $\oint \vec{E} \bullet d\vec{l} = -\frac{d\phi}{dt}$

Mutual and self inductance: $\phi_2 = M_{21} I_1$ and $M_{21} = M_{12}$

$\phi = LI$

Energy stored in a magnetic field: $W = \frac{1}{2\varepsilon_0} \int_V B^2 d\tau = \frac{1}{2} L I^2 = \frac{1}{2} \oint \vec{A} \bullet \vec{I} dl$

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl : $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

- *Schrödinger Equation.* General and time-independent:

$$
\begin{aligned}
i\hbar\frac{\partial\Psi}{\partial t} &= \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\Psi \\
H\psi &= E\psi \\
\Psi &= \psi e^{-iEt/\hbar}
\end{aligned} \tag{1}
$$

- *Formalism and Operator Algebra.* This is probably actually the easiest topic for people who have taken 143a. Some equations of use are:

$$
\begin{aligned}
\langle\alpha|\beta\rangle &= \langle\beta|\alpha\rangle^* \\
\hat{O} &= \hat{O}^\dagger \text{ if } \hat{O} \text{ is Hermitian} \\
\langle\alpha|\hat{O}|\beta\rangle &= \beta\langle\alpha|\beta\rangle \text{ if } |\beta\rangle \text{ is an eigenstate of } \hat{O} \\
&= \int_{-\infty}^{\infty} \mathrm{d}x\, \alpha^*(x)\hat{O}(x)\beta(x) \text{ in 1D} \\
&= A^\dagger \times O \times B \text{ as matrices} \\
[\hat{A},\hat{B}] &= \hat{A}\hat{B} - \hat{B}\hat{A} \\
\therefore\ [\hat{A},\hat{B}] &= -[\hat{B},\hat{A}] \\
[\hat{A}\hat{B},\hat{C}] &= [\hat{A},\hat{C}]\hat{B} + \hat{A}[\hat{B},\hat{C}]
\end{aligned} \tag{2}
$$

A Hermitian matrix has real eigenvalues; observables thus must be represented by Hermitian operators.

- *Position and momentum.*

$$
\begin{aligned}
\hat{p} &= -i\hbar\nabla \\
[\hat{x},\hat{p}] &= i\hbar \\
[\hat{f}(x),\hat{p}] &= i\hbar\frac{\mathrm{d}f}{\mathrm{d}x}
\end{aligned} \tag{3}
$$

$$\psi(x) = \langle x|\psi\rangle = \int_{-\infty}^{\infty} dp\, \langle x|p\rangle\langle p|\psi\rangle = \int_{-\infty}^{\infty} dp\, \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}\varphi(p)$$

$$R_{2s} = \left(\frac{1}{2a_o}\right)^{\frac{3}{2}}\left(2 - \frac{r}{a_o}\right)e^{-r/2a_o} \; ; \; R_{2p} = \left(\frac{1}{2a_o}\right)^{\frac{3}{2}}\left(\frac{r}{\sqrt{3}a_o}\right)e^{-\frac{r}{2a_o}} \quad ; \quad R_{1s} = \left(\frac{1}{a_o}\right)^{\frac{3}{2}} 2e^{-r/a_o}$$

$$T = \exp\left(-\frac{2}{\hbar}\int\sqrt{2m(V(x) - E)}\,dx\right) \; ; \; n! = \int_0^\infty x^n e^{-x}dx \; ; \; E = -m/2(Ze^2/4\pi\varepsilon_o\hbar n)^2$$