This test covers the topics of *Mechanics* (Topic 1) and *Electrodynamics* (Topic 2). Each topic has 4 “A” questions and 4 “B” questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.
A1 (See adjacent figure.) The thin disk of radius \( a \) has a cavity of radius \( a/2 \) centered at a point \( a/2 \) from the center of the disk and hangs at a point at top of the disk. Find the period of the physical pendulum. The mass density of the disk is \( \sigma \) (mass/unit area).

A2 A particle (mass \( m \)) is falling vertically under the influence of gravity, and a frictional force \( \vec{F} = kv \) (\( v \) is speed, \( k \) is a positive constant) is present.

\( a. \) Obtain the equation of motion (for the vertical coordinate only).

\( b. \) Integrate the equation to obtain the velocity as a function of time; find the terminal velocity.

A3 Certain neutron stars (extremely dense stars) are believed to be rotating at about 1 revolution per second. If such a star has a radius of 20 km, what must be its minimum mass so that material on its surface remains in place during the rapid rotation?

A4 A railroad freight car of mass \( 3.18 \times 10^4 \) kg collides with a stationary car. They couple together, and 27.0\% of the initial kinetic energy is transferred to thermal energy, sound, vibrations, and so on. Find the mass of the car that was initially stationary.
Mechanics Group B - Answer only two Group B questions

B1 A solid cylinder of radius 4 m, height 5 m, and volume density 2500 kg/m³ has a conical section removed from its middle. The removed cone has a base radius of 4 m and a height of 5 m; its axis of symmetry is the same as that of the initial cylinder.

What is the moment of inertia of the remaining solid about its axis of cylindrical symmetry?

B2 A solid homogeneous cylinder of radius $a$ rolls without slipping on the inside of a stationary larger cylinder of radius $R$.

a. Write down the Lagrangian function for this system.
b. Find the equation of motion.
c. Find the period of small oscillations about the stable equilibrium position.

B3 A cart of mass $M$ rolls without friction on a flat surface under the action of a force $F$. The cart supports a massless, frictionless pulley over which passes a light cord attached at either end to masses $m$ and $2m$. The coefficient of static friction between the top of the cart and the upper mass $2m$ is 0.2. What is the minimum force $F$ that must be applied to the cart in order that the upper mass $2m$ not slip on the top of the cart? $M = 10$ kg; $m = 1$ kg.
A uniform disk of radius $a$ and mass $M$ rotates about a fixed axis. A massless cord is fixed to a point on the outside circumference and leads to a massless spring (spring constant $k_1$) which is in turn fastened to a fixed point. At a radius $a/2$, another cord is fastened to a spring (spring constant $k_2$) which connects to a mass $m$.

Set up Lagrange’s equations for the disk and the weight. (Do not solve.)

The quantities $b_1$ and $b_2$ are the original lengths of the left and right springs when the whole system is at equilibrium and the disk is not rotating.

In this equilibrium state, the quantities $x_1$, $x_2$, $x_3$, and $\theta$ all equal zero.
Electrodynamics Group A - Answer only two Group A questions

A1 In a certain region of space, the free current density \( J_{\text{free}} \) is given by
\[
J_{\text{free}} = J_0 (y^2 + z^2) \hat{x}
\]
where \( J_0 = 500 \, \text{A/m}^4 \). What is the B-field vector at \((x, y, z) = (1, 1, 1) \, \text{m}\)?

A2 A capacitor comprises a long thin straight wire and a long thin, cylindrical shell of radius \( R \). The electric field outside the shell is zero everywhere; the field immediately inside the cylinder has magnitude \( E_0 \) and points toward the wire. Calculate the line charge density, \( \lambda \), on the wire.

A3 An ideal parallel-plate capacitor (plate area \( A \)) has the gap between its plates filled with two slabs of different dielectrics, with dielectric constants \( K_1 \) and \( K_2 \) (see figure). The thickness of the slabs equals the plate separation \( d \), and each slab fills half of the volume between the plates. Calculate the capacitance of this capacitor.

A4 An ohmic resistor is immersed in 200 g of water at room temperature (20°C). At time \( t = 0 \) the resistor is connected to a DC battery with \( V_0 = 120 \, \text{V} \). The graph shows the current as a function of time. Initially, the current is constant, but after 3 min. the battery starts to die, causing the current to gradually drop to zero. Assume the resistor’s resistance is temperature-independent, and that the water is thermally insulated from the environment.

a. Calculate the resistance \( R \) of the resistor.

b. Find the final temperature of the water.

Info: the specific heat of water is 
\( c_w = 4.18 \, \text{J g}^{-1} \, \text{K}^{-1} \).
**Electrodynamics Group B - Answer only two Group B questions**

**B1** A solid cylinder of radius $R$ and infinite length, with its axis of symmetry being the $z$-axis, has a polarization field $\mathbf{P} = P_0 (1 - r/R) \hat{z}$. (Here $r$ is the perpendicular distance from the $z$-axis.) In addition, it has a uniform surface charge density on its surface of $+\sigma$.

What is the electric field everywhere?

**B2** We consider the adjacent circuit. It is driven by a (sinusoidal) AC generator with 60 cycles/sec. ($f = 60$ Hz) and peak voltage amplitude $\mathcal{E} = 80$ V. We consider the circuit’s steady-state behavior.

Calculate …

a. … the current through the inductor as a function of time, $I_L(t)$

b. … the current through the resistor as a function of time, $I_R(t)$

c. … the current through the capacitor as a function of time, $I_C(t)$

d. … the amplitude of total current through the generator, $I$

e. … the circuit’s impedance $Z$

f. … the phase angle $\phi$ by which the total current $I(t)$ lags the generator’s emf

**B3** Equal but opposite currents $+I$ and $-I$ flow in two long parallel strips as shown in the figure. The currents are uniformly distributed over the strips. The width of each strip is $w$, and the distance between them is $d$ ($w >> d$).

a. Find the magnetic field between the strips. (Neglect edge effects).

b. Calculate the magnetic field energy per unit length.

c. What is the self-inductance per unit length?

**B4** A solid sphere of radius $R$, centered at the origin, is uniformly charged with charge density $\rho$. Then a small sphere is removed, making a spherical cavity centered at a point $a$ within the bigger sphere. What is the electric field vector in the cavity?
**Physical Constants**

- **speed of light** \( c = 2.998 \times 10^8 \text{ m/s} \)
- **Planck’s constant** \( h = 6.626 \times 10^{-34} \text{ J} \cdot \text{s} \)
- **Planck’s constant / \( 2\pi \)** \( h = 1.055 \times 10^{-34} \text{ J} \cdot \text{s} \)
- **Boltzmann constant** \( k_B = 1.381 \times 10^{-23} \text{ J/K} \)
- **elementary charge** \( e = 1.602 \times 10^{-19} \text{ C} \)
- **electrostatic constant** \( \varepsilon_0 = 8.854 \times 10^{-12} \text{ F/m} \)
- **magnetic permeability** \( \mu_0 = 1.257 \times 10^{-6} \text{ H/m} \)
- **gravitational constant** \( G = 6.674 \times 10^{-11} \text{ m}^3/\text{kg} \cdot \text{s}^2 \)

**EQUATIONS THAT MAY BE HELPFUL**

**ELECTROSTATICS:**

\[
\oint A \cdot dA = \frac{q_{enc}}{\varepsilon_0} \quad \text{\( \overline{\mathbf{E}} = -\nabla \mathbf{V} \)}
\]

\[
-\int_{r_1}^{r_2} \mathbf{E} \cdot d\mathbf{r} = V(r_2) - V(r_1) \quad V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}
\]

work done \( W = -\int_{\mathbf{a}}^{\mathbf{b}} q\mathbf{E} \cdot d\mathbf{r} = q\left[ V(\mathbf{b}) - V(\mathbf{a}) \right] \)

Multipole expansion:

\[
V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \left[ \frac{1}{r} \int \rho(\mathbf{r}') d\mathbf{r}' + \frac{1}{r^2} \int r' \cos(\theta') \rho(\mathbf{r}') d\mathbf{r}' + \frac{1}{r^3} \int (r')^2 \left( \frac{3}{2} \cos(\theta') - \frac{1}{2} \right) \rho(\mathbf{r}') d\mathbf{r}' + \ldots \right]
\]

where the first term is the monopole term, the second is the dipole term, the third is the quadrupole term … \( \mathbf{r} \) and \( \mathbf{r}' \) are field point and source point and \( \theta' \) is the angle between them.

\[
\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \quad \nabla \cdot \mathbf{D} = \rho_i \quad \rho_b = -\nabla \cdot \mathbf{P} \quad \sigma_b = \mathbf{P} \cdot \mathbf{n}
\]

The above are true for all dielectrics. Confining ourselves to linear, isotropic, and homogeneous (LIH) dielectrics, we also have:

\[
\mathbf{D} = \varepsilon \varepsilon_0 \mathbf{E} \quad \mathbf{P} = \chi_e \varepsilon_0 \mathbf{E} \quad \varepsilon = \varepsilon_0 (1 + \chi_e) \quad \kappa_c = \varepsilon / \varepsilon_0 \quad \chi_e = \kappa_c - 1
\]
C(dielectric) = \kappa \varepsilon C(vacuum)

Boundary conditions: \[ E_{2t} - E_{1t} = 0, \quad E_{2n} - E_{1n} = \frac{\sigma}{\varepsilon_0} \]

**MAGNETOSTATICS:**

Lorentz Force: \[ \vec{F} = q(\vec{v} \times \vec{B}) + q\vec{E} \]

Current densities: \[ I = \int \vec{J} \cdot d\vec{A}, \quad I = \int \vec{K} \cdot d\ell \]

Biot-Savart Law: \[ \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{I d\ell \times \vec{R}}{R^2} \] (\( \vec{R} \) is vector from source point to field point \( \vec{r} \)).

For surface currents: \[ \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{K} \times \vec{R}}{R^2} \, da. \]

For straight wire segment: \[ B = \frac{\mu_0 I}{4\pi s} \left[ \sin \theta_2 - \sin \theta_1 \right] \]

where \( s \) is perpendicular distance from wire.

For circular loop of radius \( R \), the \( B \)-field at a point on the axis is \[ B = \frac{1}{2} \mu_0 I \frac{R^2}{(R^2 + z^2)^{3/2}}. \]

Infinitely long solenoid: \( B \)-field inside is \( B = \mu_0 n I \) (\( n \) is number of turns per unit length).

Ampere’s law: \[ \oint \vec{B} \cdot d\ell = \mu_0 I_{\text{enclosed}}. \]

**Magnetic vector potential** \( \vec{A} \)

\[ \vec{B} = \vec{\nabla} \times \vec{A} \]

\[ \nabla^2 \vec{A} = -\mu_0 \vec{J} \quad \Rightarrow \quad \vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \, d\tau'}{r - r'} \]

For line and surface currents \[ \vec{A} = \frac{\mu_0}{4\pi} \int \frac{I \, d\ell}{r - r'} \quad \vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{K} \, d\alpha}{r - r'} \]

From Stokes' theorem \[ \oint \vec{A} \cdot d\ell = \int \vec{B} \cdot d\alpha \]

For a magnetic dipole \( \vec{m} \), \[ \vec{A}_{\text{dipole}} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^2} \]

**Magnetic dipoles**

Magnetic dipole moment of a current distribution is given by \( \vec{m} = I \int d\vec{\alpha} \).

Force on magnetic dipole \[ \vec{F} = \vec{\nabla} (\vec{m} \cdot \vec{B}) \]

Torque on magnetic dipole \[ \vec{\tau} = \vec{m} \times \vec{B} \]

\( B \)-field of magnetic dipole \[ \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{1}{r^3} \left[ 3(\vec{m} \cdot \vec{r})\vec{r} - \vec{m} \right] \]
Dipole-dipole interaction energy is 
\[ U_{DD} = \frac{\mu_0}{4\pi R^3} \left[ (\mathbf{m}_1 \cdot \mathbf{m}_2) - 3(\mathbf{m}_1 \cdot \hat{R})(\mathbf{m}_2 \cdot \hat{R}) \right], \]
where \( \hat{R} = \mathbf{r}_1 - \mathbf{r}_2 \)

**Material with magnetization \( \mathbf{M} \)**

produces a magnetic field equivalent to that of (bound) volume and surface current densities

\[ \mathbf{J}_b = \nabla \times \mathbf{M} \text{ and } \mathbf{K}_b = \mathbf{M} \times \hat{n}. \]

\[ \oint \mathbf{H} \cdot d\mathbf{l} = I_{\text{free, enclosed}} \quad \mathbf{H} = \mathbf{B} / \mu_0 - \mathbf{M} \]

For linear magnetic material \( \mathbf{M} = \chi_m \mathbf{H} \) and \( \mathbf{B} = \mu_0 (1 + \chi_m) \mathbf{H} \) or \( \mathbf{B} = \mu \mathbf{H} \)

Boundary conditions: \( B_{2n} - B_{in} = 0 \) \( B_{2\parallel} - B_{1\parallel} = \mu_0 K \)

**Maxwell’s Equations in vacuum:**

1. \( \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \) Gauss’ Law

2. \( \nabla \cdot \mathbf{B} = 0 \)

3. \( \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \) Faraday’s Law

4. \( \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \) Ampere’s Law with Maxwell’s correction

**Maxwell’s Equations in linear, isotropic, and homogeneous (LIH) media:**

1. \( \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon} \) Gauss’ Law

2. \( \nabla \cdot \mathbf{B} = 0 \)

3. \( \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \) Faraday’s Law

4. \( \nabla \times \mathbf{B} = \mu \mathbf{J}_t + \varepsilon \mu \frac{\partial \mathbf{E}}{\partial t} \) Ampere’s Law with Maxwell’s correction

Alternative way of writing Faraday’s Law: \( \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} \)

Mutual and self inductance: \( \Phi_2 = M_{21} I_1 \) and \( M_{21} = M_{12} \); \( \Phi = LI \)
Energy stored in electric, magnetic field:

\[ W = \frac{1}{2} \varepsilon_0 \int_V E^2 d\tau = \frac{Q^2}{2C} \]

\[ W = \frac{1}{2} \mu_0^{-1} \oint_{\partial V} B^2 d\tau = \frac{1}{2} LI^2 = \frac{1}{2} \oint \mathbf{A} \cdot d\mathbf{l} \]
\[ \mathbf{V} \cdot \mathbf{V} = 0 \quad : \text{Commutation} \]
\[ \mathbf{V} \cdot (\mathbf{V} \times \mathbf{V}) = 0 \quad : \text{Commutator} \]
\[ (\mathbf{V} \times \mathbf{V})' = \mathbf{V} \times (\mathbf{V} \times \mathbf{V})' \quad : \text{Commutation} \]

**Fundamental Theorems**

\[ \mathbf{V} \times (\mathbf{V} \cdot \mathbf{A}) = (\mathbf{V} \times \mathbf{A}) \times \mathbf{V} \quad (11) \]
\[ 0 = (\mathbf{f} \cdot \mathbf{A}) = (\mathbf{x} \times \mathbf{A}) \quad (10) \]
\[ 0 = (\mathbf{V} \times \mathbf{V}) \cdot \mathbf{A} \quad (6) \]

**Second Derivatives**

\[ (\mathbf{V} \cdot \mathbf{A} + \mathbf{V} + \mathbf{A} \cdot \mathbf{V}) = (\mathbf{V} \times \mathbf{V}) \cdot \mathbf{A} \quad (8) \]
\[ (\mathbf{V} \cdot \mathbf{A} + \mathbf{V} \cdot \mathbf{A}) = (\mathbf{V} \times \mathbf{A}) \cdot \mathbf{V} \quad (2) \]
\[ (\mathbf{V} \times \mathbf{V} + (\mathbf{V} \times \mathbf{A}) \times \mathbf{V}) = (\mathbf{V} \times \mathbf{A}) \cdot \mathbf{V} \quad (4) \]
\[ (\mathbf{V} \cdot \mathbf{A} + (\mathbf{V} \cdot \mathbf{A}) \cdot \mathbf{V}) = (\mathbf{V} \times \mathbf{A}) \cdot \mathbf{V} \quad (2) \]

**Product Rules**

\[ (\mathbf{V} \cdot \mathbf{V} \cdot \mathbf{C}) = (\mathbf{V} \cdot \mathbf{V}) \cdot \mathbf{V} \quad (2) \]
\[ (\mathbf{V} \times \mathbf{V}) \cdot \mathbf{C} = (\mathbf{V} \times \mathbf{V}) \cdot \mathbf{V} \quad (1) \]

**Triple Products**

\[ \mathbf{V} \cdot \mathbf{V} \cdot \mathbf{V} = \mathbf{V} \cdot \mathbf{V} \cdot \mathbf{V} \quad : \text{Commutation} \]

\[ \mathbf{V} \cdot (\mathbf{V} \times \mathbf{V}) = 0 \quad : \text{Commutator} \]

**Vector Derivatives**

\[ \left[ \frac{\partial}{\partial \mathbf{W}} - (\partial \mathbf{W}) \frac{\partial}{\partial \mathbf{W}} \right] \mathbf{V} = \mathbf{V} \mathbf{W} \quad : \text{Chain Rule} \]

\[ \frac{\partial}{\partial \mathbf{W}} \left[ \mathbf{V} \mathbf{W} \right] = \mathbf{V} + \mathbf{W} \quad : \text{Product Rule} \]

\[ \frac{\partial}{\partial \mathbf{W}} \left[ \mathbf{V} \cdot \mathbf{W} \right] = \mathbf{V} + \mathbf{W} \quad : \text{Product Rule} \]

\[ \frac{\partial}{\partial \mathbf{W}} \left[ \mathbf{V} \times \mathbf{W} \right] = \mathbf{V} \times \mathbf{W} + \mathbf{W} \times \mathbf{V} \quad : \text{Product Rule} \]

\[ \frac{\partial}{\partial \mathbf{W}} \left[ \mathbf{V} \cdot (\mathbf{V} \times \mathbf{W}) \right] = \mathbf{V} \times (\mathbf{V} \times \mathbf{W}) + \mathbf{V} (\mathbf{V} \times \mathbf{W}) \quad : \text{Product Rule} \]

\[ \frac{\partial}{\partial \mathbf{W}} \left[ \mathbf{V} \cdot (\mathbf{V} \times \mathbf{W}) \right] = \mathbf{V} \times (\mathbf{V} \times \mathbf{W}) + \mathbf{V} (\mathbf{V} \times \mathbf{W}) \quad : \text{Product Rule} \]