UNL - Department of Physics and Astronomy

Preliminary Examination - Day 2
August 16, 2013

This test covers the topics of *Quantum Mechanics* (Topic 1) and *Thermodynamics and Statistical Mechanics* (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

Quantum Mechanics Group A - Answer only two Group A questions

A1 An electron is in the state where the projection of spin on the z-axis equals $+\frac{1}{2}\hbar$. The projection of the spin on the x-axis is measured.

a. What are possible outcomes of this measurement?
b. What are the probabilities of these outcomes?
c. What is the expectation value of S_x?

A2 An electric light bulb is rated at 70 W. Assuming that it radiates light with an average wavelength of 500 nm, calculate the number of photons emitted per second.

A3 Consider the plane wave $\exp(i\mathbf{p}\cdot\mathbf{r}/\hbar)$ describing a free particle.

a. Is this a linear momentum eigenstate?
b. Is this an angular momentum eigenstate?
c. Is this a parity eigenstate?
d. If the answer to any of these questions is "no", write down the free-particle wave function which is the eigenstate of the corresponding operator.

A4 A particle of mass m is moving in a deep potential well of width L. Estimate its ground-state energy (relative to the bottom of the well) using the uncertainty principle.

Quantum Mechanics Group B - Answer only two Group B questions

B1 A particle of mass *m* is in the ground state of an infinitely deep box of width L ($V=0$ for $0<x<L$). At some instant the particle's momentum is measured. What is the probability to find its value between *p* and *p*+*dp*?

B2 The hydrogen atom is in the 2*p* state.

a. Find the expectation value of r.
b. Find the expectation value of r^2.
c. Find the most probable distance of the electron from the nucleus.
d. Find the uncertainty Δr.

B3 The hydrogen atom is in the 3*d* state. Assume that both electron orbital angular momentum $\mathbf{L}$ and spin $\mathbf{S}$ are conserved.

a. What are the possible eigenvalues of $\mathbf{J}^2$, where $\mathbf{J}=\mathbf{L}+\mathbf{S}$?
b. What are the possible eigenvalues of J_z?
c. Is the quantity $\mathbf{L}\cdot\mathbf{S}$ conserved? If it is, find its possible eigenvalues. If it isn't, find its possible expectation values.

B4 The wave function of a harmonic oscillator (mass *m*, frequency ω) at $t=0$ is given by a superposition of the normalized ground state and first excited state:

$$\psi(x)=C\left[\varphi_0(x)+\varphi_1(x)\right] .$$

a. Find the normalization constant *C* (a positive real-valued number).
b. Find the expectation value of *x* at $t=0$.
c. Find the wave function and the corresponding probability density for $t>0$.
d. Find the expectation value of *x* as a function of time.

Thermodynamics and Statistical Mechanics Group A - Answer only two Group A questions

A1 The probability that a tomato seed germinates is 60%. You plant 4 tomato seeds.

What is the probability that …

a. … exactly one seed germinates
b. … exactly three seeds germinate
c. … *at least* one seed germinates

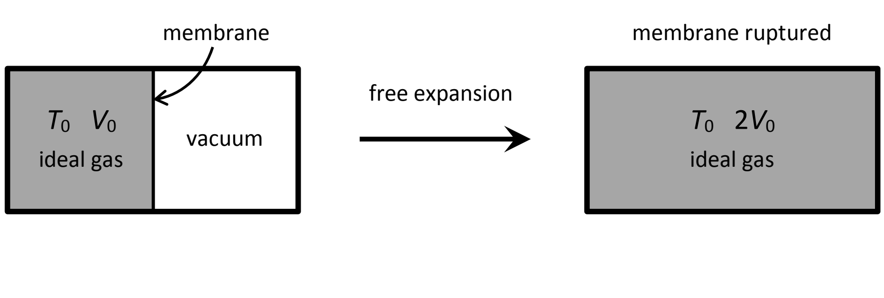
A2 What is the minimal amount of electric energy required to operate a heat pump in order to deliver 10 kJ of heat to a house, if the outside temperature is 0°C and the inside temperature is 20°C? (Assume that the inside temperature does not change in the process.)

A3 A 1 mole quantity of ideal gas is enclosed in a cylinder equipped with a piston, which is placed in a thermostat at 300 K. The gas does 650 J of work while it expands isothermally.

a. How much heat does the gas absorb in this process?
b. Where does this heat come from?
c. What is the entropy change of the gas in this process?

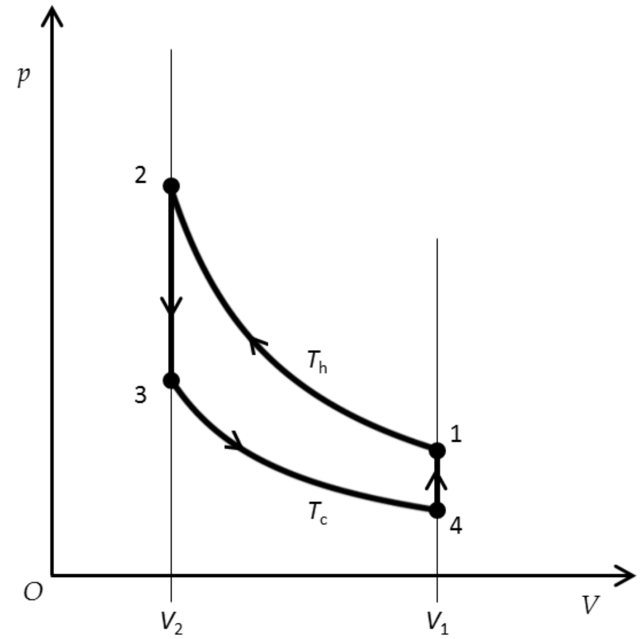
A4 See diagram below. When the mebrane suddenly ruptures, a 5 mole amount of ideal gas undergoes *free expansion* from a 1 m^3 volume to a 2 m^3 volume in a thermally insulated enclosure ($T_0 = 320$ K).

Calculate the change in entropy of the gas.



Thermodynamics and Statistical Mechanics Group B - Answer only two Group B questions

B1 The adjacent p-V diagram shows the so-called Stirling cooling cycle (refrigerator). Its working fluid is a monoatomic gas (for instance, helium). Processes 1→2 and 3→4 are isothermal: the fluid is held at constant temperature by thermal baths at temperatures T_h ("hot") and T_c ("cold"), respectively. Processes 2→3 and 4→1 are isochoric: they take place at constant volumes V_2 and V_1, respectively. The heat capacity of the working fluid at constant volume, per mole, is C_V.



Determine the heat absorbed by the fluid in …

a. … the process 1→2
b. … the process 2→3
c. … the process 3→4
d. … the process 4→1

e. Determine the net work done on the fluid per cycle.

f. Determine the "coefficient of performance" of this refrigerator. This coefficient is defined as: the amount of heat absorbed in the process 3→4, divided by the net amount of work done on the fluid in one cycle.

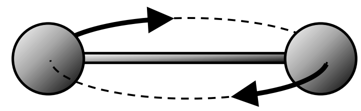
B2 In an experiment, 200 g of aluminum (with a specific heat of 900 J/kg·K) at 100°C is mixed with 50 g of water at 20°C, with the mixture thermally isolated.

a. What is the equilibrium temperature?

What are the entropy changes of…

b. … the aluminum?
c. … the water?
d. … the aluminum-water system?

B3 The quantum mechanical energy levels of a rotating dumbbell are given by



$$\mathcal{E}_\ell = B\ell(\ell+1) \qquad \text{for } \ell = 0,1,2,3,4,\ldots$$

where B is the rotational constant. For each value of ℓ there are $2\ell+1$ possible quantum states of the same energy.

a. Give the partition function *Z*.

In parts *b*. and *c*. we assume $T = 2B/k_B$.

b. Make a table of the probabilities P_ℓ that the energy of the dumbbell is $\mathcal{E}_\ell$, for the values $\ell = 0$, 1, 2, and 3 only. Explain why higher values of ℓ are barely relevant at this temperature.

c. Give the average energy of the dumbbell in terms of T.

d. What is the average energy for $T << B/k_B$?

B4 The equation of state for a *van der Waals* gas is given by

$$\left(p + \frac{a}{v^2}\right)(v-b) = RT\,,$$

where $v = V/n$ is the *molar* volume (*n* = number of moles in the gas), and *a* and *b* are positive constants. In this problem, we use the *molar* energy $\mathcal{E} = E/n$.

a. Calculate $(\partial p/\partial T)_v$ for the van der Waals gas.

b. Now calculate $(\partial \mathcal{E}/\partial v)_T$, the volume dependence of the molar energy. The answer to part *a*. may be useful.

c. Evaluate, by taking the appropriate limit, your answer for part *b.* for the limiting case of a very dilute gas. Why is this answer expected for a very dilute gas?

Physical Constants

speed of light.................. $c = 2.998 \times 10^{8}$ m/s

Planck's constant .... $h = 6.626 \times 10^{-34}$ J·s

Planck's constant / 2π.... $\hbar = 1.055 \times 10^{-34}$ J·s

Boltzmann constant ...... $k_B = 1.381 \times 10^{-23}$ J/K

elementary charge .... $e = 1.602 \times 10^{-19}$ C

electric permittivity ...... $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m

magnetic permeability ... $\mu_0 = 1.257 \times 10^{-6}$ H/m

molar gas constant.......... $R = 8.314$ J / mol·K

electrostatic constant ... $k = (4\pi\varepsilon_0)^{-1} = 8.988 \times 10^{9}$ m/F

electron mass .... $m_{el} = 9.109 \times 10^{-31}$ kg

proton mass .... $m_p = 1.673 \times 10^{-27}$ kg

1 bohr............................ $a_0 = 0.5292$ Å

1 hartree .... $E_h = 27.21$ eV

Avogrado constant ..... $N_A = 6.022 \times 10^{23}$ mol^{-1}

gravitational constant.. $G = 6.674 \times 10^{-11}$ m^3 / kg s^2

hc .... $hc = 1240$ eV·nm

EQUATIONS THAT MAY BE HELPFUL

QUANTUM MECHANICS

Stationary states of particle in infinitely deep square well of width L : $\varphi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$.

Radial wavefunctions of the hydrogen atom

n	ℓ	$R_{n\ell}$
1	0	$2\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}$
2	0	$\left(\frac{1}{2a_0}\right)^{3/2}\left(2-\frac{r}{a_0}\right)e^{-r/2a_0}$
2	1	$\left(\frac{1}{2a_0}\right)^{3/2}\frac{1}{\sqrt{3}}\frac{r}{a_0}e^{-r/2a_0}$
3	0	$2\left(\frac{1}{3a_0}\right)^{3/2}\left(1-\frac{2}{3}\frac{r}{a_0}+\frac{2}{27}(r/a_0)^2\right)e^{-r/3a_0}$
3	1	$\left(\frac{1}{3a_0}\right)^{3/2}\frac{4\sqrt{2}}{3}\left(1-\frac{1}{6}\frac{r}{a_0}\right)\frac{r}{a_0}e^{-r/3a_0}$
3	2	$\left(\frac{1}{3a_0}\right)^{3/2}\frac{2\sqrt{2}}{27\sqrt{5}}\left(\frac{r}{a_0}\right)^2 e^{-r/3a_0}$

Stationary states of harmonic oscillator for n=0 and n=1

$$\varphi_0(x) = \left(\frac{\alpha}{\pi^{1/2}}\right)^{1/2} e^{-\alpha^2 x^2/2}$$

$$\varphi_1(x) = \left(\frac{\alpha}{2\pi^{1/2}}\right)^{1/2} 2\alpha x\, e^{-\alpha^2 x^2/2}$$

where $\alpha = \left(m\omega/\hbar\right)^{1/2}$

Spin operators for a spin-1/2 particle:

Using the two eigenkets of $\hat{S}_z$ as the basis for the particle's spin state in the following notation: $S_z\begin{pmatrix}1\\0\end{pmatrix} = +\frac{1}{2}\hbar\begin{pmatrix}1\\0\end{pmatrix}$, $S_z\begin{pmatrix}0\\1\end{pmatrix} = -\frac{1}{2}\hbar\begin{pmatrix}0\\1\end{pmatrix}$, the spin operators $\hat{S}_+$, $\hat{S}_-$, and $\hat{S}_z$ are given by

$$\hat{S}_+ = \hbar\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \qquad \hat{S}_- = \hbar\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}, \qquad \hat{S}_z = \tfrac{1}{2}\hbar\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

THERMODYNAMICS

General efficiency η of a heat engine producing work $|W|$ while taking in heat $|Q_\mathrm{h}|$ is $\eta = \dfrac{|W|}{|Q_\mathrm{h}|}$.

For a Carnot cycle operating as a heat engine between reservoirs at T_h and at T_c the efficiency becomes $\eta_\mathrm{C} = \dfrac{T_\mathrm{h} - T_\mathrm{c}}{T_\mathrm{h}}$.

Clausius' theorem: $\sum_{i=1}^{N} \dfrac{Q_i}{T_i} \le 0$, which becomes $\sum_{i=1}^{N} \dfrac{Q_i}{T_i} = 0$ for a reversible cyclic process of N steps.

For adiabatic processes in an ideal gas with constant heat capacity, $PV^\gamma = \text{constant}$.

$$\frac{dP}{dT} = \frac{\lambda}{T\Delta V}$$

specific heat of water: 4186 J/(kg·K)
latent heat of ice melting: 334 J/g

$$H = E + PV \qquad F = E - TS \qquad G = F + PV \qquad \Omega = F - \mu N$$

$$dE = TdS - PdV + \mu dN \qquad dS = dE/T + PdV/T - \mu dN/T \qquad dH = TdS + VdP + \mu dN$$

$$dF = -SdT - PdV + \mu dN \qquad dG = -SdT + VdP + \mu dN \qquad d\Omega = -SdT - PdV - Nd\mu$$

$$C_V = \left(\frac{\delta Q}{dT}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V \qquad C_p = \left(\frac{\delta Q}{dT}\right)_p = T\left(\frac{\partial S}{\partial T}\right)_p \qquad TdS = C_V dT + T\left(\frac{\partial S}{\partial V}\right)_T dV$$

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial p}{\partial S}\right)_V$$
$$\left(\frac{\partial T}{\partial p}\right)_S = \left(\frac{\partial V}{\partial S}\right)_p$$
$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V$$
$$\left(\frac{\partial S}{\partial p}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_p$$

INTEGRALS

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right). \qquad \left|\tan^{-1}\left(\frac{x}{a}\right)\right| < \frac{\pi}{2}$$
$$\int \frac{x\,dx}{a^2+x^2} = \frac{1}{2}\ln(a^2+x^2)$$
$$\int \frac{dx}{x(a^2+x^2)} = \frac{1}{2a^2}\ln\left(\frac{x^2}{a^2+x^2}\right)$$
$$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right)$$
$$= -\frac{1}{ab}\coth^{-1}\left(\frac{ax}{b}\right)$$
$$= -\frac{1}{ab}\tanh^{-1}\left(\frac{ax}{b}\right). \qquad a^2x^2 < b^2$$

$$\int_0^\infty e^{-ay^2}dy = \tfrac{1}{2}(\pi/a)^{1/2}$$
$$\int_0^\infty y^2 e^{-ay^2}dy = \tfrac{1}{4}(\pi/a^3)^{1/2}$$
$$\int_0^\infty \frac{1}{1+ay^2}dy = \pi/2a^{1/2}$$
$$\int_0^\infty y^n e^{-y/a}dy = n!\,a^{n+1}$$