Mech A1

$$T = 2\pi \sqrt{\frac{I}{mg\ell}}$$

$$I_1 = \frac{1}{2} m_1 a^2 + m_1 a^2$$

$$I_2 = \frac{1}{2} m_2 \left(\frac{a}{2}\right)^2 + m_2 \left(\frac{a}{2}\right)^2 \qquad m_1 = \pi a^2 \lambda$$

$$I = I_1 - I_2 = \frac{3}{2} m_1 a^2 - \frac{3}{2} m_2 \left(\frac{a}{2}\right)^2 \qquad m_2 = \pi \left(\frac{a}{2}\right)^2 \lambda$$

$$= \frac{3}{2}(\pi a^2 \lambda) a^2 - \frac{3}{2}\left(\frac{\pi}{4} a^2 \lambda\right)\frac{a^2}{4} = \frac{45}{32}\pi \lambda a^4$$

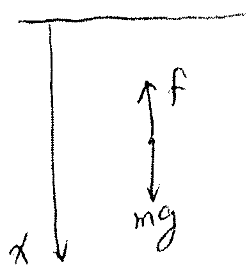
$$y_{cm} = \frac{m_1 \cdot 0 - m_2\left(\frac{a}{2}\right)}{m_1 - m_2} = \frac{-\cancel{\pi}\frac{a^2}{\cancel{4}}\cancel{\lambda} \cdot \frac{a}{2}}{\frac{3}{\cancel{4}}\cancel{\pi}\cancel{a^2}\cancel{\lambda}}$$

$$= \frac{-1}{6} a$$ ("−" just indicates that y_{cm} is below the center of the disk of radius a)

$$\ell = a + \frac{1}{6} a = \frac{7}{6} a$$

$$T = 2\pi \sqrt{\frac{\frac{45}{32}\pi \lambda a^4}{\left(\frac{3}{4}\pi a^2 \lambda\right) g \left(\frac{7}{6} a\right)}} = \pi \sqrt{\frac{45}{7}\frac{a}{g}}$$

Mech A2



$$T = \frac{1}{2}m\dot{x}^2$$

$$f = \frac{d}{dv}\left(\frac{1}{2}kv^2\right) = kv$$

since $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = Q_j$$

$$m\ddot{x} = mg - k\dot{x}$$

So: $$m\frac{dv}{dt} = mg - kv$$

$$\frac{m\,dv}{mg - kv} = dt$$

$$\frac{-m/k \cdot dv}{\frac{mg}{k} - v} = dt$$

integral: $$\ln\left(\frac{mg}{k} - v\right) = -\frac{k}{m}t + c$$

So $$v = \frac{mg}{k} - ce^{-\frac{k}{m}t}$$

Since $t=0$ $v=0$, So $c = \frac{mg}{k}$

So $$v = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right)$$

as $t \to +\infty$ $$v = \frac{mg}{k}$$

which is the maximum velocity

Mech A3

Gravitational force on some small mass m on surface is $F_g = G\dfrac{Mm}{R^2}$

Centripetal force needed for mass to stay on surface is $F_{cp} = m\omega^2 R$

For m to stay on surface we must have $F_{cp} \le F_g \quad \Rightarrow$

$$M = \frac{R^3\omega^2}{G}$$

with $R = 20000$ m and $\omega = 2\pi$ rad/s, we find $M = 4.7 \times 10^{24}$ kg.

Mech A4

Let m_F be the mass of the freight car and v_F be its initial velocity. Let m_C be the mass of the stationary car and v be the common final velocity of the two when they are coupled. Conservation of the total momentum of the two-car system leads to $m_F v_F = (m_F + m_C)v$, so $v = v_F m_F/(m_F + m_C)$. The initial kinetic energy of the system is

$$K_i = \frac{1}{2} m_F v_F^2$$

and the final kinetic energy is

$$K_f = \frac{1}{2}(m_F + m_C)v^2 = \frac{1}{2}(m_F + m_C)\frac{m_F^2 v_F^2}{(m_F + m_C)^2} = \frac{1}{2}\frac{m_F^2 v_F^2}{(m_F + m_C)}.$$

Since 27% of the original kinetic energy is lost, we have $K_f = 0.73 K_i$. Thus,

$$\frac{1}{2}\frac{m_F^2 v_F^2}{(m_F + m_C)} = (0.73)\left(\frac{1}{2} m_F v_F^2\right).$$

Simplifying, we obtain $m_F/(m_F + m_C) = 0.73$, which we use in solving for the mass of the stationary car:

$$m_C = \frac{0.27}{0.73} m_F = 0.37\, m_F = (0.37)\,(3.18 \times 10^4 \text{ kg}) = 1.18 \times 10^4 \text{ kg}.$$

Mech B1

Radius of cone is defined by $r(z)=\frac{z}{H}R$

Moment of inertia = I =

$$I=\rho\int_0^H dz\int_{r(z)}^R r'^2 2\pi r'dr'=2\pi\rho\int_0^H dz\int_{r(z)}^R r'^3 dr'=2\pi\rho\int_0^H dz\left[\tfrac{1}{4}r'^4\right]_{r(z)}^R=2\tfrac{1}{4}\pi\rho\int_0^H dz\left[R^4-r(z)^4\right]=$$
$$=\tfrac{1}{2}\pi\rho\left\{\int_0^H R^4 dz-\int_0^H r(z)^4 dz\right\}=\tfrac{1}{2}\pi\rho\left\{R^4H-\frac{R^4}{H^4}\left[\tfrac{1}{5}z^5\right]_0^H\right\}=\tfrac{1}{2}\pi\rho\left\{R^4H-\frac{R^4}{H^4}\tfrac{1}{5}H^5\right\}=\tfrac{1}{2}\pi\rho R^4\left\{H-\tfrac{1}{5}H\right\}=$$
$$=\tfrac{1}{2}\pi\rho R^4\,\tfrac{4}{5}H=\tfrac{2}{5}\pi\rho R^4H$$

Using $R=4, H=5, \rho=2500$ kg/m^3 we find

$$I=\tfrac{2}{5}\pi\rho R^4H=\tfrac{2}{5}\pi\times2500\times4^4\times5=2\pi\times2500\times4^4=4.02\times10^6\ \text{kg}\cdot\text{m}^2$$

(Note:)

Volume

$$\text{vol.}=\int_0^H dz\int_{r(z)}^R 2\pi r'dr'=2\pi\int_0^H dz\int_{r(z)}^R r'dr'=2\pi\int_0^H dz\left[\tfrac{1}{2}r'^2\right]_{r(z)}^R=2\tfrac{1}{2}\pi\int_0^H dz\left[R^2-r(z)^2\right]=\pi\left\{\int_0^H R^2dz-\int_0^H r(z)^2dz\right\}=$$
$$=\pi\left\{R^2H-\int_0^H\frac{z^2}{H^2}R^2dz\right\}=\pi\left\{R^2H-\frac{R^2}{H^2}\int_0^H z^2dz\right\}=\pi\left\{R^2H-\frac{R^2}{H^2}\tfrac{1}{3}H^3\right\}=\pi\left\{R^2H-\tfrac{1}{3}R^2H\right\}=\tfrac{2}{3}\pi R^2H$$

Mass density = mass/volume = $\rho=\frac{\text{mass}}{\text{vol.}}=\frac{m}{\tfrac{2}{3}\pi R^2H}=\tfrac{3}{2}\frac{m}{\pi R^2H}$

```
(* define the cone *)
```

$$\text{cone}[z\_] := \frac{z}{H} R$$

```
(* calculate the integral of r^2 over the whole object *)
(* integral over r *)
```

$$\text{Integrate}\left[2 \pi r r^2, \{r, a, R\}\right]$$

$$-\frac{a^4 \pi}{2} + \frac{\pi R^4}{2}$$

```
(* followed by integral over z *)
```

$$\text{r2intg} = \text{Integrate}\left[-\frac{(\text{cone}[z])^4 \pi}{2} + \frac{\pi R^4}{2}, \{z, 0, H\}\right]$$

$$\frac{2}{5} H \pi R^4$$

```
(* calculate the integral of 1 over
 the whole object to get the volume *)
(* integral over r *)
```

$$\text{Integrate}[2 \pi r , \{r, a, R\}]$$

$$-a^2 \pi - \pi R^2$$

```
(* followed by integral over z *)
```

$$\text{Volume} = \text{Integrate}\left[-(\text{cone}[z])^2 \pi + \pi R^2, \{z, 0, H\}\right]$$

$$\frac{2}{3} H \pi R^2$$

```
(* now calculate mom. of inertia*)
```

$$\frac{\text{mass}}{\text{Volume}} * \text{r2intg}$$

$$\frac{3 \text{ mass } R^2}{5}$$

```
(* insert numerical values *)
```

$$2500. \times \frac{2}{5} 5 \pi 4^4$$

$$4.02124 \times 10^6$$

Mech B2

The kinetic energy of the cylinder can be decomposed into two parts, one involving the translation of the center of mass, and the other involving rotation about the center of mass. Let $\dot{\theta}$ be the angular velocity of the cylinder. The kinetic energy T can be written:

$$T = \frac{I_o}{2}\dot{\theta}^2 + \frac{m}{2}(R - a)^2\dot{\delta}^2 \tag{1}$$

where

$$I_o = \frac{1}{2}ma^2$$

for a solid cylinder. The constraint of rolling without sliding is $a\dot{\theta} = (R - a)\dot{\delta}$. Hence (1) becomes

$$T = \frac{m}{4}(R - a)^2\dot{\delta}^2 + \frac{m}{2}(R - a)^2\dot{\delta}^2$$

$$= \frac{3}{4}m(R - a)^2\dot{\delta}^2 \tag{2}$$

The potential energy is

$$V = -(R - a)mg \cos \delta \tag{3}$$

$(V = 0 \text{ for } \delta = 90°)$

The Lagrangian function is

$$L = T - V = \frac{3m}{4}(R - a)^2\dot{\delta}^2 + mg(R - a) \cos \delta . \tag{4}$$

Substituting (4) into Lagrange's equation we obtain the equation of motion

$$\frac{3}{2}(R - a)\ddot{\delta} + g \sin \delta = 0. \tag{5}$$

For small δ we have $\sin \delta \simeq \delta$, and (5) becomes

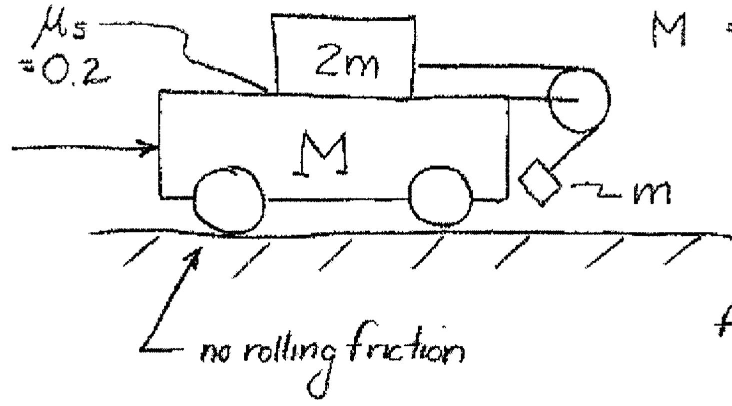
$$\ddot{\delta} + \frac{2g}{3(R - a)}\delta = 0$$

from which we get

$$\omega = \sqrt{\frac{2g}{(3(R - a))}} .$$

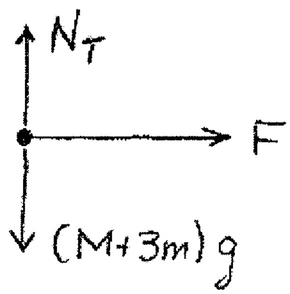
$m = 1\,kg$

$M = 10\,kg$



What is the minimum value of F that keeps 2m from slipping?

entire system

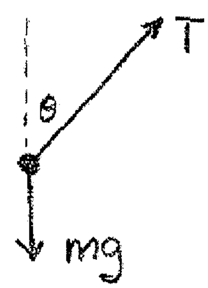


x) $F_{x,tot} = m_{TOT}\, a_x$

$$\boxed{F = (M+3m)a} \quad (1)$$

unknowns : F, a

m



x) $F_{x,tot} = m a_x$

$$\boxed{T\sin\theta = ma} \quad (2)$$

unknowns : T, θ, a

y) $F_{y,tot} = m a_y$

$T\cos\theta - mg = 0 ; \quad \boxed{T\cos\theta = mg} \quad (3)$

unknowns : T, θ

Let b_1, b_2 be the original lengths of springs 1 and 2 when the whole system is at equilibrium and the disc is not rotating. If θ is the angle the disc has turned away from the equilibrium position, then spring 1 is stretched by a distance $x_1 = a\theta$. If spring 2 is stretched by a distance x_2, the weight is lowered by a distance $x_3 = \frac{a}{2}\theta + x_2$. The kinetic energy of the system is:

Kinetic energy of the disc $= \frac{1}{2} I\dot{\theta}^2$

$$= \frac{m}{4} a^2 \dot{\theta}^2 \qquad (1)$$

Kinetic energy of the weight $= \frac{1}{2} m\dot{x}_3^2$

$$= \frac{m}{2}\left(\frac{a}{2}\dot{\theta} + \dot{x}_2\right)^2 \qquad (2)$$

The potential energy of the system is

$$V = \frac{1}{2} ka^2\theta^2 + \frac{1}{2} kx_2^2 - mgx_3$$

$$= \frac{1}{2} ka^2\theta^2 + \frac{1}{2} kx_2^2 - mg\left(\frac{a}{2}\theta + x_2\right) \qquad (3)$$

Therefore the Lagrangian function is

$$L = T - V$$

$$= \frac{m}{4} a^2\dot{\theta}^2 + \frac{m}{2}\left(\frac{a^2}{4}\dot{\theta}^2 + a\dot{\theta}\dot{x}_2 + \dot{x}_2^2\right) - \frac{k}{2}\left(a^2\theta^2 + x_2^2\right) + mg\left(\frac{a}{2}\theta + x_2\right). \qquad (4)$$

The Lagrange's equations are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} - \frac{\partial L}{\partial x_2} = 0 \text{ and } \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0. \qquad (5)$$

Substituting (4) into (5), we can immediately get two equations of motion.

E&M A1

$$\mathbf{J}_{\text{free}} = J_0(y^2 + z^2)\hat{\mathbf{x}}$$

Thus current density has AXIAL SYMMETRY about the x-axis.

$$\mathbf{B} = B(\rho)\hat{\boldsymbol{\phi}} \qquad (\rho = \sqrt{x^2 + z^2})$$

$$\oint \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I_{\text{encl.}} \qquad (*)$$

where the AMPERIAN LOOP of the integral is a circle of radius $\rho = \sqrt{2}\,m$ lying in the plane $x = 1$, centered on the x-axis.

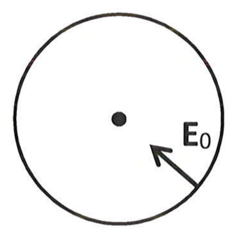
Thus, the left-hand side of (*) = $B(\rho)2\pi(\sqrt{2}\,m)$.

$$I_{\text{encl.}} = \int_0^{\sqrt{2}m} J(\rho)dA = \int_0^{\sqrt{2}m} J_0\rho^2 2\pi\rho d\rho = 2\pi J_0 \int_0^{\sqrt{2}m} \rho^3 d\rho = \frac{2\pi J_0(\sqrt{2})^4}{4} = 250\pi \times 4 = 3142 \text{ A}$$

Thus $B(\sqrt{2}\,m) = \dfrac{3142 \text{ A} \times 4\pi \times 10^{-7} \text{ H/m}}{2\pi\sqrt{2}\,m} = 4.44 \times 10^{-4} \text{ T} = 4.44 \text{ gauss.}$

Direction (by inspection) = $\dfrac{\hat{\mathbf{z}} - \hat{\mathbf{y}}}{\sqrt{2}}$, so $\boxed{\mathbf{B} = 3.14 \text{ gauss } (\hat{\mathbf{z}} - \hat{\mathbf{y}})}$

A2 A capacitor comprises a long thin straight wire and a long thin, cylindrical shell of radius R. The electric field outside the shell is zero everywhere; the field immediately inside the cylinder has magnitude E_0 and points toward the wire.



Calculate the line charge density, λ, on the wire.

ANSWER:

A) From Gauss' Law, we find field of line alone is $E = \dfrac{\lambda}{2\pi\varepsilon_0 r}$.

B) Outer shell does not contribute to fld inside

C) Therefore, $E_0 = \dfrac{\lambda}{2\pi\varepsilon_0 R} \Rightarrow \lambda = 2\pi\varepsilon_0 R E_0$

Easy way

This is equivalent to two separate capacitors (each of half the area A) in parallel. One has capacitance $C_1 = K_1 \frac{\varepsilon_0(\frac{1}{2}A)}{d}$; the other $C_2 = K_2 \frac{\varepsilon_0(\frac{1}{2}A)}{d}$.

Net capacitance is then $C = C_1 + C_2 = K_1 \frac{\varepsilon_0(\frac{1}{2}A)}{d} + K_2 \frac{\varepsilon_0(\frac{1}{2}A)}{d} = \frac{1}{2}(K_1 + K_2)\frac{\varepsilon_0 A}{d}$

Hard way

Charge cap with charge Q and calculate resulting voltage, then find $C = Q/V$.

Charge distributes itself such that electric fields left/right become equal.

Say free surface charge on left part of plate is σ_1, on right side, σ_2 .

Field in left part is $E_1 = \frac{1}{K_1}\frac{\sigma_1}{\varepsilon_0}$, in right part, $E_2 = \frac{1}{K_2}\frac{\sigma_2}{\varepsilon_0}$.

Fields must be equal: $\frac{1}{K_1}\frac{\sigma_1}{\varepsilon_0} = \frac{1}{K_2}\frac{\sigma_2}{\varepsilon_0} \quad \Rightarrow \quad \frac{\sigma_1}{K_1} = \frac{\sigma_2}{K_2}$ (Eq. #1)

We have $\sigma_1 \frac{1}{2}A + \sigma_2 \frac{1}{2}A = Q \quad \Rightarrow \quad \sigma_1 + \sigma_2 = \frac{2Q}{A}$ (Eq. #2)

Solving for σ_1, σ_2 , we get $\sigma_1 = \frac{2Q}{A}\frac{K_1}{K_1 + K_2}$ and $\sigma_2 = \frac{2Q}{A}\frac{K_2}{K_1 + K_2}$

$$E_1 = E_2 = \frac{2Q}{\varepsilon_0 A}\frac{1}{K_1 + K_2} \quad \Rightarrow \quad V = Ed = \frac{2Q}{\varepsilon_0 A}\frac{d}{K_1 + K_2} \quad \Rightarrow \quad C = \frac{Q}{V} = \frac{\varepsilon_0 A}{d}\frac{1}{2}(K_1 + K_2)$$

E and M A4 - SOLUTIONS

Expiring Battery

a.

Initially, the current is 2 ampere, and the battery has a voltage of 120 V. Therefore, $V_B = IR \quad \Rightarrow \quad R = \frac{V_B}{I} = \frac{120}{2} = 60\ \Omega$.

b.

The power (in J/s = W) with which heat is developed in the resistor is $P(t) = I^2(t)R$. The total amount of heat Q added to the water is then $Q = \int_0^\infty P(t')dt' = R\int_0^\infty I^2(t')dt' = \underbrace{R\int_0^{180\text{ s}} I^2(t')dt'}_{\equiv Q_1} + \underbrace{R\int_{180}^{240\text{ s}} I^2(t')dt'}_{\equiv Q_2}$. The first term is simple because the current is constant: $Q_1 = 60 \times \int_0^{180\text{ s}} 4 \cdot dt' = 240 \times 180 = 43.2\text{ kJ}$. To calculate the second term, we need a function describing the linear drop of the current between 180 s and 240 s, from 2 amps to 0 amps. This function is $I(t) = 8 - t/30$. Then

$$Q_2 = R\int_{180}^{240\text{ s}} I^2(t)dt = 60\int_{180}^{240\text{ s}} (8 - t/30)^2 dt = 60\left[-10(8 - t/30)^3\right]_{t=180}^{240} =$$
$$60\left[\left\{-10(8 - 240/30)^3\right\} - \left\{-10(8 - 180/30)^3\right\}\right] = 60 * 80 = 4800 = 4.8\text{ kJ}$$

The total heat added to the water is thus $Q = Q_1 + Q_2 = 43.2 + 4.8 = 48\text{ kJ}$. The temperature increase is $\Delta t = \frac{Q}{mc_w} = \frac{48{,}000}{200 * 4.18} = 57\text{ K}$. The final temperature of the water is thus $t_f = t_i + \Delta t = 20°\text{C} + 57°\text{C} = 77°\text{C}$

E&M B1

$$\mathbf{P} = P_0(1 - r/R)\hat{\mathbf{z}}$$
$$\sigma_{\text{bound}} = \mathbf{P}\cdot\hat{\mathbf{n}} = 0$$
$$\rho_{\text{bound}} = -\nabla\cdot\mathbf{P}$$

$\mathbf{P} = (P_\rho, P_\phi, P_z) = (0, 0, P_z)$ in cylindrical coordinates.

In cylindrical coordinates, $-\nabla\cdot\mathbf{P} = -\dfrac{\partial P_z}{\partial z} = 0 = \rho_{\text{bound}}$, thus $\mathbf{E}(r \le R) = \mathbf{0}$.

$$\lambda\ell = 2\pi R\ell\sigma$$
$$\lambda = 2\pi R\sigma$$
$$\lambda\ell = 2\pi r\ell\varepsilon_0 E(r); \quad E(r \ge R) = \frac{\lambda}{2\pi r\varepsilon_0} = \frac{2\pi R\sigma}{2\pi r\varepsilon_0} = \frac{R\sigma}{r\varepsilon_0}$$

$$\boxed{\mathbf{E}(r \ge R) = \frac{R\sigma}{r\varepsilon_0}\hat{\mathbf{r}}}$$

E&M B2

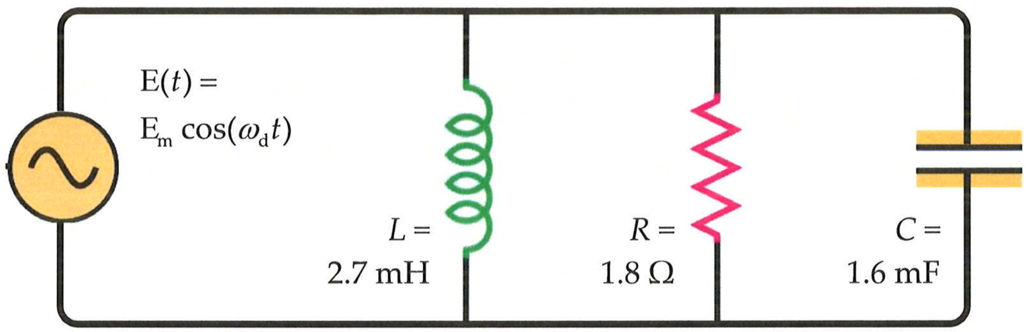


Fig. 1 - *A parallel LRC circuit.*

SOLUTIONS

(we use $X_L = \omega L, \quad X_R = R, \quad X_C = 1/\omega C$)

$I_L = \frac{\mathrm{E_m}}{X_L} = \frac{\mathrm{E_m}}{\omega L} = \frac{80}{377 * 2.7\times10^{-3}} = 79 \text{ A}$ lags $\mathrm{E_m}$ by 90°

$I_R = \frac{\mathrm{E_m}}{R} = \frac{80}{1.8} = 44 \text{ A}$ in phase with $\mathrm{E_m}$

$I_C = \frac{\mathrm{E_m}}{X_C} = \omega C\mathrm{E_m} = 377 * 1.6\times10^{-3} * 80 = 48 \text{ A}$ leads $\mathrm{E_m}$ by 90°

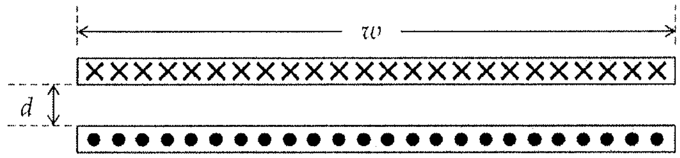
Amplitude of total current $I_{\text{tot}} = \sqrt{(I_L - I_C)^2 + I_R^{\,2}} = \sqrt{(79-48)^2 + 44^2} = 54 \text{ A}$.

We must have $I_{\text{tot}} = \frac{\mathrm{E_m}}{Z} \Rightarrow Z = \frac{\mathrm{E_m}}{I_{\text{tot}}} = \frac{80}{54} = 1.5\ \Omega$.

We have $|\varphi| = \arctan\left(\frac{|I_L - I_C|}{I_R}\right) = \arctan(31/44) = 35°$.

Because $I_L > I_C$ the total current lags the emf so $\varphi = +35°$.

E&M B3



(a)

Field between plates due to one plate $2B d\ell = \mu_0 \frac{I}{w} d\ell \quad \Rightarrow \quad B = \frac{\mu_0 I}{2w}$.

Field due to both plates $B = \frac{\mu_0 I}{w}$.

(b)

$W = \frac{B^2}{2\mu_0} w d \Delta\ell$, where $\Delta\ell$ is the length in the direction of the currents

$$\frac{W}{\Delta\ell} = \frac{\mu_0 I^2 d}{2w}$$

(c)

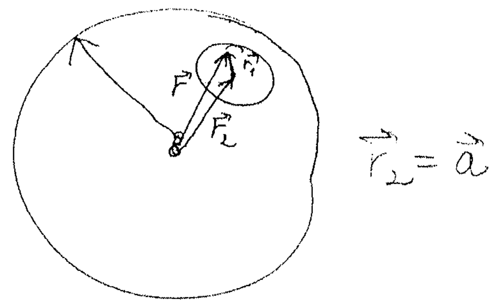
$$W = \tfrac{1}{2} L I^2 \quad \Rightarrow \quad \frac{W}{\Delta\ell} = \tfrac{1}{2}(L/\Delta\ell) I^2 \quad \Rightarrow \quad L/\Delta\ell = \frac{\mu_0 d}{W}$$

E&M B4

Calculate first field due to uniformly charged sphere

$$E 4\pi r^2 = \frac{\rho}{\varepsilon_0}\tfrac{4}{3}\pi r^3 \quad \Rightarrow \quad E = \frac{\rho}{3\varepsilon_0} r \quad \Rightarrow \quad \mathbf{E}(\mathbf{r}) = \frac{\rho \mathbf{r}}{3\varepsilon_0}$$

Then use the superposition principle to calculate field at the position **r** in the figure.



$\mathbf{E}(\mathbf{r}) = \mathbf{E}_x + \mathbf{E}_{\text{cav.}}$ where $\mathbf{E}_x$ is unknown field, and $\mathbf{E}_{\text{cav.}}$ is the field due to charge filling the cavity

$$\mathbf{E}_{\text{cav.}}(\mathbf{r}) = \frac{\rho \mathbf{r}_1}{3\varepsilon_0}$$

$\mathbf{E}_x = \mathbf{E}(\mathbf{r}) - \mathbf{E}_{\text{cav.}} = \frac{\rho}{3\varepsilon_0}(\mathbf{r} - \mathbf{r}_1) = \frac{\rho \mathbf{r}_2}{3\varepsilon_0}$ where $\mathbf{r}_2 = \mathbf{a}$ is the position of the center of the cavity.