UNL - Department of Physics and Astronomy

Preliminary Examination - Day 2
August 15, 2014

This test covers the topics of *Thermodynamics and Statistical Mechanics* (Topic 1) and *Mechanics* (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY

Thermodynamics and Statistical Mechanics Group A - Answer only two Group A questions

A1 The compressibility of a gas is defined as $K = -\frac{1}{V}\frac{dV}{dp}$ and depends on the process. Find the compressibility of an ideal gas with a given constant $\gamma = C_p / C_V$ in

a. an isothermal process;
b. an adiabatic process.

A2 Two balloons of equal volume are filled, at the same pressure, with hydrogen and helium, respectively.

Which one of them has larger vertical net force? By what factor is it larger?

A3 A piece of string is cut in two at a point selected at random.

Find the probability $\mathrm{P}(x)$ that the longer piece is at least x times larger than the shorter piece (for $x > 1$).

A4 For a diatomic ideal gas near room temperature, what fraction of the heat supplied is used to produce work, if the gas is expanded at constant temperature?

Thermodynamics and Statistical Mechanics Group B - Answer only two Group B questions

B1 Two bodies with different initial temperatures T_1 and T_2 are brought into contact while being insulated from their surroundings. The heat capacities of the two bodies are equal.

Find the total entropy change accompanying the equilibration of the system and show explicitly that it is always positive.

B2 Let the mean-squared velocity of molecules in a gas be $\overline{v}$ and their temperature be T. Starting from the general form of the Gibbs distribution, show that in thermodynamic equilibrium $\log \overline{v} = a + b \log T$ and find the numerical coefficient b.

B3 A body is brought from temperature T_i to temperature T_f by placing it in contact with a reservoir at temperature T_f. Assuming that the heat capacity of the body, C, is temperature independent

a. compute the total entropy change of the system (body + reservoir).

b. demonstrate that it is positive whether $T_i > T_f$ or $T_i < T_f$. (*Hint:* $x - 1 > \ln x$ *for* $x > 0$ *and* $x \neq 1$).

The same body is now brought from temperature T_i to temperature T_f by placing it in contact with a series of N reservoirs at temperatures $T_i + \Delta T$, $T_i + 2\Delta T$, … $T_i + N\Delta T \equiv T_f$.

c. Calculate the total entropy change of the system for this process for the case when $N \to \infty$ for fixed T_i and T_f.

B4 (*Amoebas are single-celled organisms.*) A population starts with a single amoeba. For this one and for the generations thereafter, there is a probability p that an individual amoeba will split to create two amoebas (otherwise it dies out without offspring).

What is the probability that the family tree of the original amoeba will go on forever?
Hint: Consider the probability that at least one of the two amoebas after the initial splitting produces an infinite family tree.

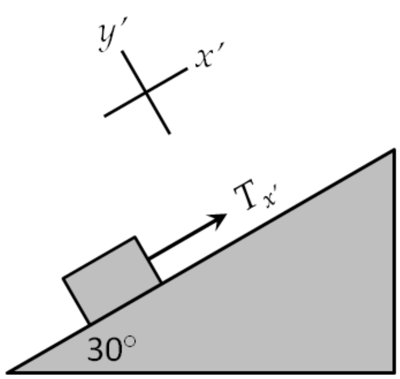
<u>Mechanics Group A</u> <u>- Answer only two Group A questions</u>

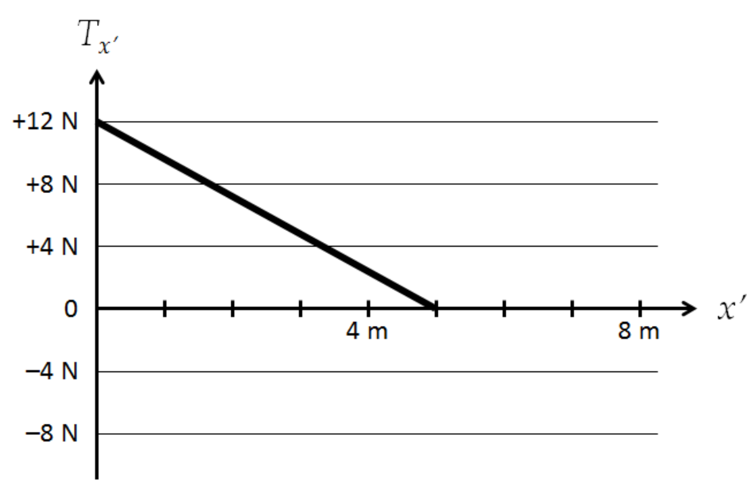
A1 A particle of unit mass moving along a straight line experiences a damping force $-\dot{x}$ and a force that depends on the particle's displacement from the origin as $+x - x^3$.

a. Find the points of equilibrium of the particle.
b. For each point, find out whether it is a point of the stable or unstable equilibrium.

A2 A 0.8 kg block is pulled up a $30°$ frictionless incline. It starts from rest at the incline's base ($x' = 0$). The pull on the block is provided by a light string which is parallel to the incline. The tension in the string decreases linearly with distance that the block has moved up the incline from the bottom until the block has moved 5 m. At this point, the string goes slack, and the block moves under the action of gravity and the incline's normal force only. The graph of the string's tension *vs*. distance the block has moved (x') is shown below.

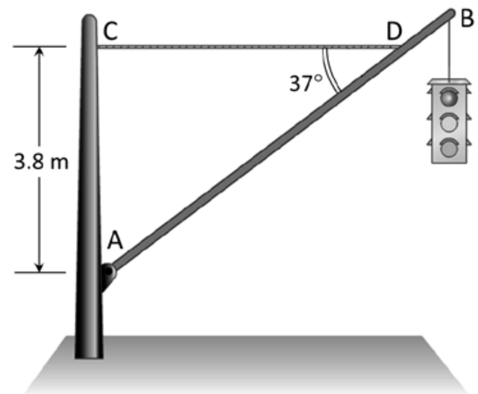
a. Plot the component of the gravitational force on the block parallel to x' as a function of x' on the graph below.
b. Using graphical integration, calculate the work done on the block by the string.
c. What is the maximum value of x' that the block reaches?





A3 A traffic light of mass 12 kg is suspended from a uniform support beam of mass 53 kg and length $\overline{AB} = 7.2\ \text{m}$ as shown. Note that $\overline{AD} < \overline{AB}$.

Calculate the tension in the massless cable CD.



A4 A spherical planet with radius 6000 km has a mass density that decreases linearly with distance from the center. The density is $15 \times 10^3\ \text{kg/m}^3$ at the center and $2.0 \times 10^3\ \text{kg/m}^3$ at the surface.

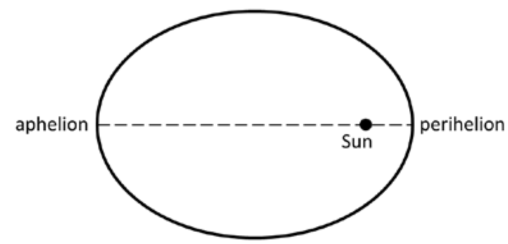
Calculate the acceleration due to gravity at the surface of this planet.

Mechanics Group B - Answer only two Group B questions

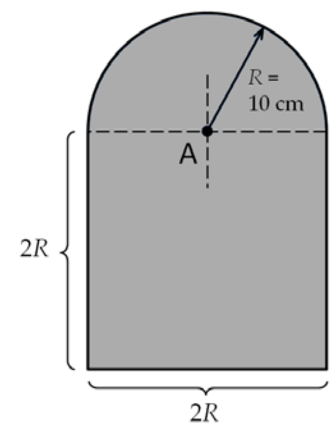
B1 A comet is moving along an elliptical orbit whose perihelion is r_p and aphelion r_a. The Sun's mass is M.

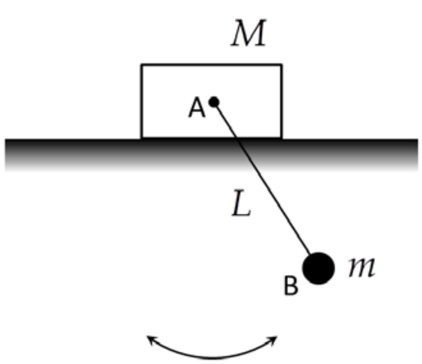
a. Find the comet's reduced angular momentum L/m (m is the comet's mass).

b. Find the comet's reduced total energy E/m.



B2 What is the moment of inertia of the 5 kg plane laminate shown in the adjacent figure about an axis perpendicular to the page that intersects point A?



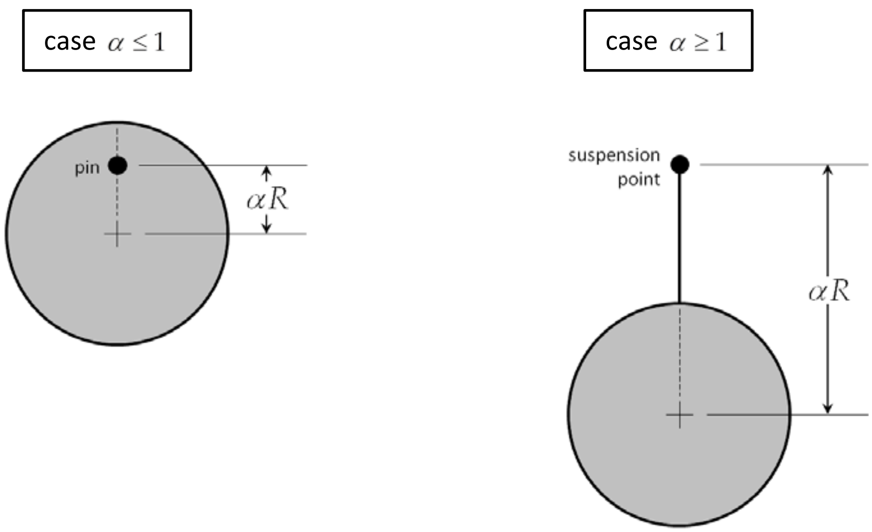


B3 A "two-mass pendulum", shown in the adjacent figure, consists of a mass M that moves horizontally without friction on a surface. A massless rod AB of length L is attached by a frictionless pivot to the mass M and is connected at the bottom to a small mass m.

Devise a suitable coordinate system with which to describe this apparatus, and write down its equations of motion.

B4 A uniform solid disk of radius R = 50 cm and mass M = 15 kg is suspended to form a physical pendulum. Under the action of the Earth's gravitational field, it swings back and forth with a period T. The suspension point is a distance αR above the disk's center, with $0 \le \alpha \le \infty$. If $\alpha \le 1$, the suspension is effected by drilling a hole through the disk at $r = \alpha R$ and having the disk pivot back and forth on a frictionless pin inserted through the hole. If $\alpha \ge 1$, the disk is suspended from a light string of length $(\alpha - 1)R$ attached to the disk's edge.

What value of α minimizes T?



Physical Constants

speed of light .... $c = 2.998 \times 10^{8}$ m/s

Planck's constant .... $h = 6.626 \times 10^{-34}$ J·s

Planck's constant / 2π.... $\hbar = 1.055 \times 10^{-34}$ J·s

Boltzmann constant ...... $k_B = 1.381 \times 10^{-23}$ J/K

elementary charge .... $e = 1.602 \times 10^{-19}$ C

electric permittivity ...... $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m

magnetic permeability ... $\mu_0 = 1.257 \times 10^{-6}$ H/m

molar gas constant.......... $R = 8.314$ J / mol·K

Avogrado constant .... $N_A = 6.022 \times 10^{23}$ mol^{-1}

electrostatic constant ... $k = (4\pi\varepsilon_0)^{-1} = 8.988 \times 10^{9}$ m/F

electron mass .... $m_{el} = 9.109 \times 10^{-31}$ kg

electron rest energy...... 511.0 keV

Compton wavelength .. $\lambda_C = h / m_{el}c = 2.426$ pm

proton mass .... $m_p = 1.673 \times 10^{-27}$ kg $= 1836 m_{el}$

1 bohr .... $a_0 = \hbar^2 / ke^2 m_{el} = 0.5292$ Å

1 hartree (= 2 rydberg) ... $E_h = \hbar^2 / m_{el} a_0^{\,2} = 27.21$ eV

gravitational constant ... $G = 6.674 \times 10^{-11}$ m^3/ kg s^2

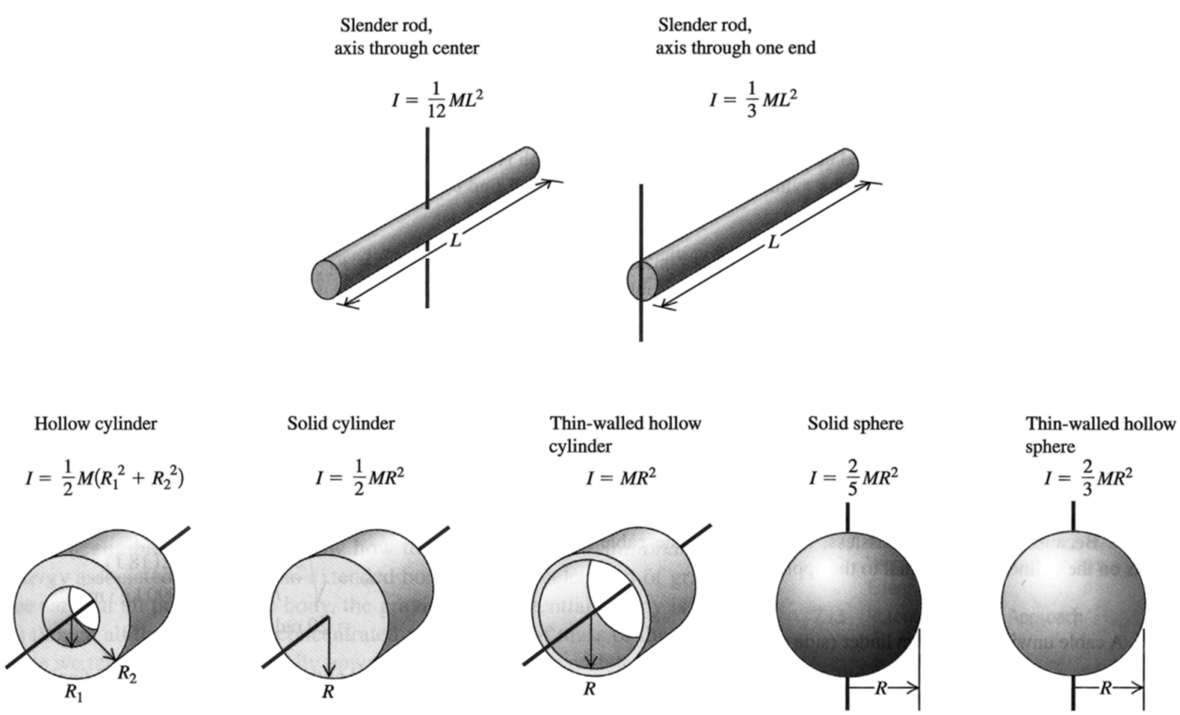
hc .... $hc = 1240$ eV·nm

Equations That May Be Helpful

MECHANICS

Gravitational acceleration on Earth: $g = 9.81$ m/s^2

Moments of Inertia of Various Bodies:



THERMODYNAMICS

General efficiency η of a heat engine producing work $|W|$ while taking in heat $|Q_h|$ is $\eta = \frac{|W|}{|Q_h|}$.

For a Carnot cycle operating as a heat engine between reservoirs at T_h and at T_c the efficiency becomes $\eta_C = \frac{T_h - T_c}{T_h}$.

Clausius' theorem: $\sum_{i=1}^{N} \frac{Q_i}{T_i} \le 0$, which becomes $\sum_{i=1}^{N} \frac{Q_i}{T_i} = 0$ for a reversible cyclic process of N steps.

For adiabatic processes in an ideal gas with constant heat capacity, $PV^{\gamma} = \text{constant}$.

$$\frac{dP}{dT} = \frac{\lambda}{T\Delta V}$$

specific heat of water:	4186 J/(kg·K)
latent heat of ice melting:	334 J/g

$$H = E + PV \qquad F = E - TS \qquad G = F + PV \qquad \Omega = F - \mu N$$

$$dE = TdS - PdV + \mu dN \qquad dS = dE/T + PdV/T - \mu dN/T \qquad dH = TdS + VdP + \mu dN$$

$$dF = -SdT - PdV + \mu dN \qquad dG = -SdT + VdP + \mu dN \qquad d\Omega = -SdT - PdV - Nd\mu$$

$$C_V = \left(\frac{\delta Q}{dT}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V \qquad C_p = \left(\frac{\delta Q}{dT}\right)_p = T\left(\frac{\partial S}{\partial T}\right)_p \qquad TdS = C_V dT + T\left(\frac{\partial S}{\partial V}\right)_T dV$$

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial p}{\partial S}\right)_V$$

$$\left(\frac{\partial T}{\partial p}\right)_S = \left(\frac{\partial V}{\partial S}\right)_p$$

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V$$

$$\left(\frac{\partial S}{\partial p}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_p$$

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence : $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

Curl : $\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$

Laplacian : $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$

Divergence : $\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$

Curl : $\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$

Laplacian : $\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence : $\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$

Curl : $\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$

Laplacian : $\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

INTEGRALS

$f(x)$	$\int_0^\infty f(x)\,dx$
e^{-ax^2} ……………	$\frac{\sqrt{\pi}}{2\sqrt{a}}$
xe^{-ax^2} ……………	$\frac{1}{2a}$
$x^2 e^{-ax^2}$ ……………	$\frac{\sqrt{\pi}}{4a^{3/2}}$
$x^3 e^{-ax^2}$ ……………	$\frac{1}{2a^2}$
$x^4 e^{-ax^2}$ ……………	$\frac{3\sqrt{\pi}}{8a^{5/2}}$
$x^5 e^{-ax^2}$ ……………	$\frac{1}{a^3}$
$x^6 e^{-ax^2}$ ……………	$\frac{15\sqrt{\pi}}{16a^{7/2}}$

$$\int_0^\infty \frac{1}{1+ay^2}\,dy = \pi / 2a^{1/2}$$

$$\int_0^\infty y^n e^{-y/a}\,dy = n!\,a^{n+1}$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right), \qquad \left|\tan^{-1}\left(\frac{x}{a}\right)\right| < \frac{\pi}{2}$$

$$\int \frac{x\,dx}{a^2+x^2} = \frac{1}{2}\ln(a^2+x^2)$$

$$\int \frac{dx}{x(a^2+x^2)} = \frac{1}{2a^2}\ln\left(\frac{x^2}{a^2+x^2}\right)$$

$$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right)$$

$$= -\frac{1}{ab}\coth^{-1}\left(\frac{ax}{b}\right)$$

$$= -\frac{1}{ab}\tanh^{-1}\left(\frac{ax}{b}\right), \qquad a^2x^2 < b^2$$