**E&M A1**

Consider the following bizarre circuit shown below. Initially, S1 is closed, and remains closed until a steady-state situation is reached. The battery potential is 5V, L = 4 mH, and R = 2Ω. What is the energy stored in the inductor? Now, S1 is opened, and, simultaneously, S2 is closed. How much power is being dissipated in the resistor 5 msec after S2 is closed?

\[
I_{\text{Initial}} = \frac{\mathcal{E}}{R} = \frac{5}{2} \sqrt{\frac{L}{2}} = 2.5 \text{ A}
\]

\[
U = \frac{1}{2} L I^2 = \frac{25}{8} \times 4 \times 10^{-3} = 12.5 \text{ mJ}
\]

\[
I = I_0 e^{-\left(\frac{R}{L}\right) t}
\]

\[
\text{at} \ 5 \text{ msec} = 2.5 \text{ A} \times \exp\left(-\frac{2}{4 \times 10^{-3} \times 5 \times 10^{-3}}\right) = 205 \text{ mA}
\]

\[
P = i^2 R = 84 \text{ mW}
\]

---

**E&M A2**

Consider a long current-carrying wire extending to infinity along the + and - axes. Near the origin, it is given a semi-circular deformation of radius a as shown. The wire carries a current I from right to left. What is the magnitude and direction of the B-field at the origin?

\[
\vec{B}(0) = \frac{\mu_0 I}{4\pi} \int_0^{180^\circ} \, d\theta = \frac{\mu_0 I}{4\pi a^2} \int_0^\pi \, \theta d\theta = \frac{\mu_0 I a}{4\pi} \Rightarrow \frac{\mu_0 I}{4\pi} (\hat{\theta} \times \vec{a})
\]
Consider a Gaussian cylinder of radius 2.5 cm, concentric with the conductors.

\[
\frac{1}{\varepsilon_0} \lambda (2 \text{cm}) \lambda = 2\pi \varepsilon_0 \lambda E_0
\]

\[
\frac{1}{\varepsilon_0} \lambda (2 \text{cm}) = -2\pi \times 1.5 \times 10^{-2} \text{ m} \times 7 \text{ N/C}
\]

\[
8.98 \times 10^9 \lambda (2 \text{cm}) = -0.75 \times 10^{-2} \times 7
\]

\[
\lambda (2 \text{cm}) = \frac{-7.5 \times 10^{-3} \times 7}{8.98 \times 10^9} = -5.85 \times 10^{-12} \text{ C/m}
\]

\[
15 = X - 5.85
\]

\[
\lambda (2.5 \text{ cm}) = +20.85 \text{ pC/m}
\]
Where $\varepsilon = vBL = IR \Rightarrow I_0 = \frac{vBL}{R}$ and $F_0 = ILB = \frac{vB^2L^2}{R}$.

For the loop pulled through the region of magnetic field,

Where $\varepsilon = vBL = IR \Rightarrow I_0 = \frac{vBL}{R}$ and $F_0 = ILB = \frac{vB^2L^2}{R}$.
E&M B1

\[ \Phi = q \frac{\alpha}{r} \left(1 + \frac{\alpha}{2r}\right) \]

(a) When \( r = 0 \), \( \Phi \) has a singularity at \( r = 0 \),

\[ r \to 0, \quad \nabla^2 \Phi = \nabla^2 (\Phi r) = 4\pi q \delta(r) \]

It is interpreted as a point charges situates at \( r = 0 \) with a magnitude of \( 4\pi q \).

(b) \( r \neq 0 \)

\[ \nabla^2 \Phi = -4\pi \rho(r) \]

By spherical coordinates

\[ \nabla^2 \Phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) = \frac{q}{\alpha n} e^{-\frac{r}{\alpha}} \]

\[ S = -\frac{q}{\alpha n} e^{-\frac{r}{\alpha}} \]

The charge distribution is continuous except at \( r = 0 \). With a maximum value \( -\frac{q}{\alpha n} \), the charge density decay exponentially. The total orbital electronic charge \( Q \)

\[ Q = \int_{V} \rho(r) dV = -\frac{q}{\alpha n} \int_{0}^{\infty} r^2 e^{-\frac{2r}{\alpha}} dr = -\frac{q}{\alpha} P(\alpha) \]

i.e. there is a charge equal and opposite to the total electronic charge of the atom at \( r = 0 \), which is the nucleus.
E&M B2

Suppose the electric field inside a large piece of isotropic dielectric is \( E_0 \), so that the electric displacement is \( D_0 = \varepsilon_0 E_0 + P \). Now a long, thin, needle-shaped cavity is hollowed out of the material. This cavity runs parallel to \( P \). We assume the polarization is “frozen in”, so it doesn’t change when the cavity is made. We also assume the cavity is small enough that \( P \), \( E_0 \), and \( D_0 \) are essentially uniform in the solid.

a. Find the electric field vector \( E \) at the center of the cavity in terms of \( E_0 \) and \( P \).

b. Find the electric displacement vector \( D \) at the center of the cavity in terms of \( D_0 \) and \( P \).

Part a. The boundary condition for the \( E \) field is \( E_{\text{above}} - E_{\text{below}} = \frac{\sigma}{\varepsilon_0} \hat{n} \), but here \( \sigma = \sigma_b = P \cdot \hat{n} = 0 \), so \( E_{\text{in needle}} = E_{\text{solid}} = E_0 \).

Part b. We have no polarization in the cavity, so \( P_{\text{in needle}} = 0 \).

Hence, \( D_{\text{in needle}} = \varepsilon_0 E_{\text{in needle}} + P_{\text{in needle}} = \varepsilon_0 E_0 = D_0 - P \)
The potential at point $P$ (\( \mathbf{E}_P \))

\[
\mathbf{E}_P = -\frac{q}{r_1} - \frac{q}{r_2} \quad \ldots (1)
\]

where $r_1 = \sqrt{r^2 + d^2 - 2rd\cos\theta}$; $r_2 = \sqrt{r^2 + d^2 + 2rd\cos\theta}$

\[
\hat{n} \cdot (\mathbf{E}_2 - \mathbf{E}_1) = 4\pi\sigma
\]

\[
\hat{n} \cdot (\nabla \phi) = -4\pi\sigma
\]

Use spherical polar co-ordinates with the $z$ axis passing through $q$ and $q'$. Since $\mathbf{E}$ is not a function of $\phi$, we obtain

\[
\sigma = -\frac{1}{4\pi} \hat{n} \cdot \left( \mathbf{E}_r \frac{\partial \mathbf{E}}{\partial r} + \frac{\partial \mathbf{E}}{\partial \theta} \frac{\partial \hat{\phi}}{\partial \theta} \right) \quad (2)
\]

Substitute $\theta = \frac{\pi}{2}$ and $r = \rho$, i.e., on the conducting plane where $\rho$ is the radius on the plane with the origin as the center, Eq. (2) becomes:

\[
\sigma = -\frac{q_d}{2\pi (\rho^2 + d^2)^{\frac{3}{2}}} \quad (3)
\]

By Coulomb's Law of force between its image,

\[
\mathbf{F} = \frac{q q'}{(2d)^2} \hat{k} = -\frac{q^2}{(2d)^2} \hat{k}
\]
Consider a sphere of radius $R$ with a constant uniform magnetization $M$. The magnetic field inside the sphere is given by $\mathbf{B} = \frac{2}{3} \mu_0 \mathbf{M}$.

\begin{enumerate}
\item Calculate the surface current density at any point on the surface.
\item Calculate the tangential component of the $B$ field just outside the sphere.
\item Calculate the normal component of the $B$ field just outside the sphere.
\end{enumerate}

**SOLUTION**

We’ll need

\[
\mathbf{\hat{x}} = \sin \theta \cos \phi \mathbf{\hat{r}} + \cos \theta \cos \phi \mathbf{\hat{\theta}} - \sin \phi \mathbf{\hat{\phi}}
\]

\[
\mathbf{\hat{y}} = \sin \theta \sin \phi \mathbf{\hat{r}} + \cos \theta \sin \phi \mathbf{\hat{\theta}} + \cos \phi \mathbf{\hat{\phi}}
\]

\[
\mathbf{\hat{z}} = \cos \theta \mathbf{\hat{r}} - \sin \theta \mathbf{\hat{\theta}}
\]

We also have $\mathbf{\hat{z}} \times \mathbf{\hat{r}} = (\cos \theta \mathbf{\hat{r}} - \sin \theta \mathbf{\hat{\theta}}) \times \mathbf{\hat{r}} = -\sin \theta \mathbf{\hat{\theta}} \times \mathbf{\hat{r}} = -\sin \theta (-\mathbf{\hat{\phi}}) = \sin \theta \mathbf{\hat{\phi}}$

Define these quantities:

- $\mathbf{B}^{(0)}_{\perp}$ = inside field perpendicular
- $\mathbf{B}^{(0)}_{\parallel}$ = inside field parallel
- $\mathbf{B}^{(0)}_{\perp}$ = outside field perpendicular
- $\mathbf{B}^{(0)}_{\parallel}$ = outside field parallel

\[
\mathbf{B}^{(0)} = \frac{2}{3} \mu_0 \mathbf{M} = \frac{2}{3} \mu_0 \mathbf{M} \mathbf{\hat{z}} = \frac{2}{3} \mu_0 \mathbf{M} (\cos \theta \mathbf{\hat{r}} - \sin \theta \mathbf{\hat{\theta}}) = \frac{2}{3} \mu_0 \mathbf{M} \cos \theta \mathbf{\hat{r}} - \frac{2}{3} \mu_0 \mathbf{M} \sin \theta \mathbf{\hat{\theta}}
\]

\[
\mathbf{K}_b = \mathbf{K}_{b,\parallel} + \mathbf{K}_{b,\perp} = \mathbf{M} \times \mathbf{\hat{n}} = \mathbf{M} \mathbf{\hat{z}} \times \mathbf{\hat{r}} = \mathbf{M} \sin \theta \mathbf{\hat{\phi}}
\]

NOTE: this is $\parallel$

so $\mathbf{K}_{b,\parallel} = \mathbf{M} \sin \theta \mathbf{\hat{\phi}}$ and $\mathbf{K}_{b,\perp} = 0$
\[
\begin{align*}
\frac{\mathbf{B}^{(0)} - \mathbf{B}^{(i)}}{\mu_0} &= \mathbf{K} \times \hat{n}_{\parallel} \quad \text{so} \quad \mathbf{B}^{(0)} = \mathbf{B}^{(i)} = \frac{2}{3} \mu_0 M \cos \theta \hat{r} \\
\frac{\mathbf{B}^{(0)} - \mathbf{B}^{(i)}}{\mu_0} &= \mathbf{K} \times \hat{n} = M \sin \theta \hat{\phi} \times \hat{r} = M \sin \theta \hat{\Theta} \quad \Rightarrow \\
\Rightarrow \quad \mathbf{B}^{(0)} &= \mathbf{B}^{(i)} + \mu_0 M \sin \theta \hat{\Theta} = -\frac{2}{3} \mu_0 M \sin \theta \hat{\Theta} + \mu_0 M \sin \theta \hat{\Theta} = \frac{1}{3} \mu_0 M \sin \theta \hat{\Theta}
\end{align*}
\]
Quantum A1

1. intensity \( I = UC \)

\[
u = \frac{I}{C} \quad \lambda = \frac{U}{h} = \frac{UC}{h} = \frac{I}{hC^2}
\]

\[
= \frac{5 \times 10^4 \text{ W}}{m^2} \cdot 5 \times 10^{-7} \text{ m}
\]

\[
\frac{6.63 \times 10^{-34} \text{ J s} \cdot 9 \times 10^{16} \text{ m}^2/s^2}{6.63 \times 10^{-34} \text{ J s} \cdot 9 \times 10^{16} \text{ m}^2/s^2} = 0.42 \times 10^{15} \text{ photons/m}^2
\]

Quantum A2

2. \( n_{\text{min}} = l + 1 = 7 \)

\[
E = -\frac{13.6 \text{ eV}}{h^2} = -\frac{13.6 \text{ eV}}{49} = -0.28 \text{ eV}
\]

Quantum A3

3. wave function

\[
\psi(x) = \exp \left\{ -\frac{i}{\hbar} \sqrt{2m(V_0 - E)} x \right\}
\]

penetration depth

\[
= \frac{\hbar}{\sqrt{2m(V_0 - E)}} = \frac{1.055 \times 10^{-34}}{\sqrt{2 \times 9.1 \times 10^{-31} \times 15.1 \times 10^{-19}}}
\]

\[
= 0.05 \times 10^{-9} \text{ m} = 0.05 \text{ nm}
\]
4. \( E \propto \mu \) where \( \mu = \text{the reduced mass} \)

\[
\mu = \frac{m_u m_p}{m_u + m_p} = \frac{207 \times 1836}{207 + 1836} m_e
\]

\[
= 186 m_e
\]

\[
E = -13.6 \text{eV} \times 186 = 2530 \text{eV}
\]

\[\text{Hard}\]

Quantum B1

5. \( \lambda' - \lambda = \lambda_c (1 - \cos \Theta) \)

(a) \( \lambda' - \lambda \) is max when \( \Theta = 180^\circ \)

(b) \( \lambda' - \lambda = 2 \lambda_c \)

\[
\lambda' = \lambda + 2 \lambda_c = \frac{hc}{E} + 2 \lambda_c
\]

\[
E' = \frac{hc}{\lambda'} - \frac{hc}{\lambda}
\]

\[
K_e = E - E' = E - \frac{hc}{\lambda'} = E - \frac{hc}{\lambda_c + \lambda_c}
\]

\[
= E - \frac{1}{E + E_0}
\]

where \( E_0 = \frac{hc}{\lambda_c} = \frac{hc}{\frac{hc}{m} = mc^2} \)

\[
K_e = \frac{1 + \frac{E_0}{E} - 1}{1 + \frac{E_0}{E}} = \frac{E}{1 + \frac{E_0}{E}} = \frac{200}{1 + \frac{200}{200}}
\]

\[
= 56 \text{ keV}
\]
If the particle is localized within the region limited by $1 \times 1$, it creates the momentum uncertainty $\frac{h}{x}$ and the min possible momentum $\frac{p}{x}$.

$$E = \frac{p^2}{2m} + Ax^4$$

minimum with respect to $x$:

$$-\frac{h^2}{8mA^2} + 4Ax^3 = 0$$

$$x^3 = \frac{h}{(4ma)^{1/3}} \quad x = \frac{h^{1/3}}{(4ma)^{1/6}}$$

$$p = \frac{h^{2/3}}{(4ma)^{1/6}}$$

$$E = \frac{h^2}{2m} \frac{(4ma)^{1/3}}{h^{2/3}} + a \frac{h^{1/3}}{(4ma)^{1/6}} = \frac{h^{4/3}a^{1/3}}{m^{1/3}}$$
\[ \psi(x, o) = e^{i \left( \frac{1}{2} (\mathcal{H} - E_1) \right)} \]

(a) \[ e = \frac{1}{\sqrt{2}} \]

(b) \[ \psi(x, t) = \frac{1}{\sqrt{2}} \left[ \psi_1(x) e^{-iE_1t/\hbar} + \psi_2(x) e^{-iE_2t/\hbar} \right] \]

\[ E = \frac{\hbar^2}{2m} \int |\psi(x, t)|^2 \frac{\text{d}x}{dx} \]

\[ \langle E \rangle = \frac{1}{2} \int \left[ \psi_1^*(x) e^{iE_1t/\hbar} + \psi_2^*(x) e^{iE_2t/\hbar} \right] \psi_1(x) e^{-iE_1t/\hbar} + \psi_2(x) e^{-iE_2t/\hbar} \text{d}x \]

\[ \mathcal{H} \left[ \psi_1 e^{-iE_1t/\hbar} + \psi_2 e^{-iE_2t/\hbar} \right] \]

\[ = \frac{1}{2} (E_1 + E_2) \]

since \[ \mathcal{H} \psi_1 = E_1 \psi_1 \]

\[ \mathcal{H} \psi_2 = E_2 \psi_2 \]

(c) \[ \int |\psi(x, t)|^2 \text{d}x = \]

\[ \frac{1}{2} \int \left[ |\psi_1(x)|^2 + |\psi_2(x)|^2 \right]^2 + 2 \text{Re} \left\{ \psi_1(x) \psi_2(x) e^{i(E_1 - E_2)/\hbar} \right\} \text{d}x \]

\[ = \int \psi_1(x) \psi_2(x) \text{d}x \]

\[ = \int \psi_1(x) \psi_2(x) \text{d}x \cos \cot \theta = A \cos \cot \theta \]

\[ \cot \theta = \frac{E_1 - E_2}{\hbar} \]

\[ \langle x \rangle = 0 \quad \text{for start state} \]

since \[ |\psi_1|^2 \] and \[ |\psi_2|^2 \] are even

(d) energies can be found from the de Broglie

\[ p = \frac{h}{2\alpha} \]

\[ E = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{8ma^2} = \frac{n^2 \hbar^2}{2ma^2} \]

or give them energies
\[ \omega = \frac{E_2 - E_1}{\hbar} = \frac{\pi^2 \hbar^2}{2ma^2} \left( l^2 - 1^2 \right) = \frac{3\pi^2 \hbar}{2ma^2} \]

\[ T = \frac{\omega}{2\pi} = \frac{4ma^2}{\omega \hbar} \]

Classical time \[ \frac{2a}{v} = 2a \sqrt{\frac{m}{2\epsilon}} \]

Substitute \[ E = \frac{E_1 + E_2}{2} = \frac{5a^4 \hbar^2}{4ma^2} \]

\[ T_{cl} = 2a \sqrt{\frac{2ma^2}{5a^4 \hbar^2}} = \frac{2a}{\hbar} \sqrt{\frac{2}{5}} \]

\[ \frac{T}{T_{cl}} = \frac{4/5}{2 \sqrt{\frac{2}{5}}} = \frac{2}{3} \sqrt{\frac{5}{2}} = \frac{\sqrt{10}}{3} = 1.05 \]

Quantum B4

\[ f \sim A r^2 e^{-r/\alpha_0} \]

\[ P(r) = \rho^2 r^2 = A r^6 e^{-2r/\alpha_0} \]

(a) \[ \langle r \rangle = \frac{\int r^7 e^{-2r/\alpha_0} dr}{\int r^6 e^{-2r/\alpha_0} dr} = \left( \frac{\alpha_0}{2} \right)^3 \frac{7!}{6!} \]

\[ = \frac{3}{2} \alpha_0 \cdot 7 = 10.5 \alpha_0 \]

(b) \[ \frac{d\rho}{dr} = 0 \implies 6r^5 - \frac{2}{3} \alpha_0 r^6 = 0 \implies r = 9 \alpha_0 \]

(c) \[ l = 2, \quad L = \hbar \sqrt{2.3} = \hbar \sqrt{6} \]

(d) \[ E = -\frac{\hbar^2 \omega}{9} = -1.51 \text{eV} \]