

## THERMO A1

Problem: probability (easy)

After the first book is selected, there are 16 books left of the other two types. Total, there are 23 books left. Thus, the probability of pulling a different kind is  $\frac{16}{23}$

## THERMO A2

Problem: First Law (easy)

The work for expansion at constant pressure is

$$W = \int_A^B p dV = \int_{V_0}^{2V} p dV = pV_0 = \underline{RT_0}$$

The increase of the internal energy is

$$\Delta U = C_V \Delta T = \frac{3}{2} R \Delta T = \frac{3}{2} p_0 V = \frac{3}{2} pV_0 = \frac{3}{2} RT_0$$

The heat absorbed by the gas is

$$Q = \Delta U + W = \underline{\underline{\frac{5}{2} RT_0}}$$

## (SOLUTIONS FOR A1, A2, B1 GIVEN ELSEWHERE)

### THERMO A3

Using the first law of thermodynamics, write down the expression for the differential of internal energy  $dE$  in a reversible process for a system that can only do work by expansion, in variables  $V$  and  $S$ . Show how this leads to the definition of the Gibbs free energy  $G$  in variables  $T$  and  $P$  and write the expression for  $dG$  in these variables. Write down the Maxwell relation following directly from this last expression for  $dG$ .

Solution:  $\delta Q = dE + \delta W$ ,  $dE = pdV$ ,  $\delta Q = TdS$ ,  $dE = -pdV + TdS$ .

$G = E + pV - TS$ ,  $dG = VdP - SdT$ .

$(\partial V/\partial T)_P = -(\partial S/\partial P)_T$ .

### THERMO A4

A reversible Carnot cycle is used in an electrically-powered heat pump to maintain a 20°C room temperature in a building while the outside air temperature is 4°C. Heat flow through the surface of the building at these conditions amounts to 10 kW. What is the minimum power that this heat pump must consume from the electric power source?

Solution: Rate of heat released by heat pump to the building (denote it  $P_2$ ) should balance the outflow through its surface. Let  $P_1$  be the rate of heat absorption from the outside air. Then  $P_1/T_1 = P_2/T_2$  and  $P_1 = P_2(T_1/T_2)$ . The electric power  $P$  consumed by the heat pump is at least  $(P_2 - P_1) = P_2(1 - T_1/T_2) = 10 \text{ kW} (1 - 277/293) = 546 \text{ W}$ .

### THERMO B2

Suppose that a one-dimensional unbiased random walker starts out at the origin  $x = 0$  at  $t = 0$  and takes unit length steps at regular intervals. The probability of a step to the right is  $p$ . Define a random variable, the *first-passage time*, equal to the number of steps  $n$  it will take for the walker to *first* reach  $x = +1$ . Find the probabilities for  $n$  to be equal 1, 2, ..., 7 for a particular realization of this random walk.

Solution:

First,  $n$  obviously can't be odd, so  $P(2)=P(4)=P(6)=0$ .

$n=1$  means the walker steps to the right on the first step. Therefore,  $P(1) = 1/2$ .

$n=3$  can only happen in one way: LRR (left, right, right). So  $P(3)=1/8$ .

$n=5$  can happen as LLRRR or LRLRR. So  $P(5)=2/2^5=1/16$ .

$n=7$ : LLLRRRR, LLR(LRRR,RLRR), LR(LLRRR,LRLRR). So  $P(7)=5/2^7$ .

Problem: entropy (hard)

The temperature gradient in the bar is  $\frac{T_H - T_c}{L}$ ,  
 temperature at the cross section at a distance  $x$   
 from the cold end is  $T(x) = T_c + \frac{T_H - T_c}{L} x$ .

The bar is adiabatically removed and comes  
 to a stationary state, we have:

$$\int_0^L \rho c_p (T(x) - T_f) dx = 0$$

Substitute  $T(x)$ , integrate, and solve for  $T_f$ :

$$T_f = \frac{T_H + T_c}{2}$$

Next:  $c_p = T \left( \frac{\partial S}{\partial T} \right)_p$

$$\Rightarrow \Delta S = c_p \rho A \int_0^L dx \int_{T(x)}^{\frac{(T_H + T_c)}{2}} \frac{dT}{T} = C_p \left[ 1 + \ln T_f \right.$$

$$\left. + \frac{T_c}{T_H - T_c} \ln T_c - \frac{T_H}{T_H - T_c} \ln T_H \right]$$

where  $C_p = c_p V = c_p \rho A L$

OR  
 equivalently  $= C_p \left( 1 + \ln \frac{T_f}{T_c} - \frac{T_H}{T_H - T_c} \ln \frac{T_H}{T_c} \right)$

Full calculation of the integral:

$$\begin{aligned} \Delta S &= c_p p A \int_0^L dx \int_{T(x)}^{T_H} \frac{dT}{T} = c_p p A \int_0^L dx \left[ \ln T \right]_{T(x)}^{T_H} \\ &= c_p p A \int_0^L dx \ln \frac{T_H}{T(x)} = c_p p A \int_0^L dx \ln \frac{T_H}{T_c + \frac{T_H - T_c}{L} x} \\ &= c_p p A \int_0^L dx \left[ \ln \left( \frac{T_H}{T_c} \frac{1}{1 + \frac{T_H - T_c}{T_c} \frac{x}{L}} \right) \right] \\ &= c_p p A \int_0^L dx \left[ \ln \frac{T_H}{T_c} - \ln \left( 1 + \frac{T_H - T_c}{T_c} \frac{x}{L} \right) \right] \\ &= c_p p A \left[ \ln \left( \frac{T_H}{T_c} \right) x \Big|_0^L - \int_0^L dx \ln(1 + ax) \right] \end{aligned}$$

where  $a = \frac{T_H - T_c}{T_c L}$ . Substitute  $y = ax$   
 $dx = \frac{dy}{a}$

$$\begin{aligned} &= \underbrace{c_p p A L}_{C_p} \ln \frac{T_H}{T_c} - p p A \int_0^{aL} \frac{dy}{a} \ln(1+y) = \\ &= C_p \ln \frac{T_H}{T_c} - \frac{c_p p A}{a} \left[ (y+1)(\ln(y+1) - 1) \right]_0^{aL} \\ &= C_p \ln \frac{T_H}{T_c} - \frac{c_p p A}{a} \left[ (aL+1)(\ln(aL+1) - 1) - 1 \cdot (0-1) \right] \\ &= C_p \ln \frac{T_H}{T_c} - \frac{c_p p A}{a} \left[ (aL+1) \ln(aL+1) - aL \right] \\ aL &= \frac{T_H - T_c}{T_c L} L = \frac{T_H - T_c}{T_c}, \quad aL+1 = \frac{T_H - T_c}{T_c} + 1 = \frac{T_H}{T_c} \end{aligned}$$

$$\begin{aligned} \Delta S &= C_p \ln \frac{T_H}{T_c} - \frac{c_p p A}{T_H - T_c} \left[ \frac{T_H}{T_c} \ln \frac{T_H}{T_c} - \frac{T_H - T_c}{T_c} \right] \\ &= C_p \ln \frac{T_H}{T_c} - \frac{C_p}{T_H - T_c} \left[ T_H \ln \frac{T_H}{T_c} - (T_H - T_c) \right] \\ &= C_p \ln \frac{T_H}{T_c} - C_p \left[ \frac{T_H}{T_H - T_c} \ln \frac{T_H}{T_c} - 1 \right] = C_p \left[ 1 + \ln \frac{T_H}{T_c} - \frac{T_H}{T_H - T_c} \ln \frac{T_H}{T_c} \right] \end{aligned}$$

### THERMO B3

For a classical gas in equilibrium at temperature  $T$ , calculate the statistical averages  $\langle v_x v_y \rangle$  and  $\langle v_x^2 v_y^2 \rangle$ . Here  $v_x, v_y$  are the Cartesian components of the velocity of a given particle of mass  $m$ .

Solution: The x and y components of the velocity are uncorrelated because they enter the Hamiltonian in a separable way, so  $\langle v_x v_y \rangle = \langle v_x \rangle \langle v_y \rangle = 0$ . Similarly,  $\langle v_x^2 v_y^2 \rangle = \langle v_x^2 \rangle \langle v_y^2 \rangle$ . But  $m \langle v_x^2 \rangle / 2 = kT/2$  (by the equipartition theorem), and  $\langle v_x^2 \rangle = kT/m$ . Therefore,  $\langle v_x^2 v_y^2 \rangle = (kT/m)^2$ .

### THERMO B4

A 3 kg bronze brick heated to 100°C is placed in a thermally insulated vessel with 1 kg of ice, which is initially at 0°C. The whole system is then allowed to reach equilibrium.

- a) What is the final temperature of the system?
- b) How much ice has thawed?
- c) What is the total change of entropy of the system?

*Reference data:* Specific heat of water: 4186 J/(kg·K). Specific heat of ice: 2050 J/(kg·K). Specific heat of bronze: 435 J/(kg·K). It takes 334 joules to melt 1 gram of ice.

## Mechanics A1

A wheel is free to rotate about its fixed axle. A spring is attached to one of its spokes a distance  $r$  from the axle, as shown in Fig.

(a) Assuming that the wheel is a hoop of mass  $m$  and radius  $R$ , what is the angular frequency  $\omega$  of small oscillations of this system in terms of  $m$ ,  $R$ ,  $r$ , and the spring constant  $k$ ? What is  $\omega$  if (b)  $r = R$  and (c)  $r = 0$ ?

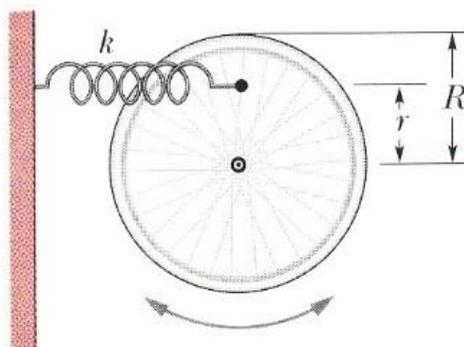


Fig.

## Mechanics A1 Solution

(a) The rotational inertia of a hoop is  $I = mR^2$ , and the energy of the system becomes

$$E = \frac{1}{2} I \omega^2 + \frac{1}{2} kx^2$$

and  $\theta$  is in radians. We note that  $r\omega = v$  (where  $v = dx/dt$ ). Thus, the energy becomes

$$E = \frac{1}{2} \left( \frac{mR^2}{r^2} \right) v^2 + \frac{1}{2} kx^2$$

which looks like the energy of the simple harmonic oscillator discussed in §15-4 if we identify the mass  $m$  in that section with the term  $mR^2/r^2$  appearing in this problem. Making this identification, Eq. 15-12 yields

$$\omega = \sqrt{\frac{k}{mR^2/r^2}} = \frac{r}{R} \sqrt{\frac{k}{m}}$$

(b) If  $r = R$  the result of part (a) reduces to  $\omega = \sqrt{k/m}$ .

(c) And if  $r = 0$  then  $\omega = 0$  (the spring exerts no restoring torque on the wheel so that it is not brought back towards its equilibrium position).

## Mechanics A2

A projectile is fired vertically from Earth's surface with an initial speed of 10 km/s. Neglecting air drag, how far above the surface of Earth will it go?

### Mechanics A2 solution

Energy conservation for this situation may be expressed as follows:

$$K_1 + U_1 = K_2 + U_2$$
$$\frac{1}{2}mv_1^2 - \frac{GmM}{r_1} = \frac{1}{2}mv_2^2 - \frac{GmM}{r_2}$$

where  $M = 5.98 \times 10^{24}$  kg,  $r_1 = R = 6.37 \times 10^6$  m and  $v_1 = 10000$  m/s. Setting  $v_2 = 0$  to find the maximum of its trajectory, we solve the above equation (noting that  $m$  cancels in the process) and obtain  $r_2 = 3.2 \times 10^7$  m. This implies that its *altitude* is  $r_2 - R = 2.5 \times 10^7$  m.

# MECHANICS A3 SOLUTION

## Easy Mechanics Problem # 2

A uniform string of mass 1.5 grams is put under a tension of 55 N between two pegs 2m apart. The string is vibrating in the second harmonic above its fundamental frequency, with an amplitude of 2 mm. For this motion, what is the highest speed any part of the rope attains?

$$\omega = \frac{\pi n}{L} \sqrt{\frac{T}{\mu}} = \frac{3\pi}{L} \sqrt{\frac{T}{\mu}}$$

$$v = \sqrt{\frac{T}{\mu}}$$

$$\lambda = \frac{2}{3}L$$

$$\lambda f = v; f = \frac{v}{\lambda} = \sqrt{\frac{T}{\mu}} \frac{3}{2}L^{-1}$$

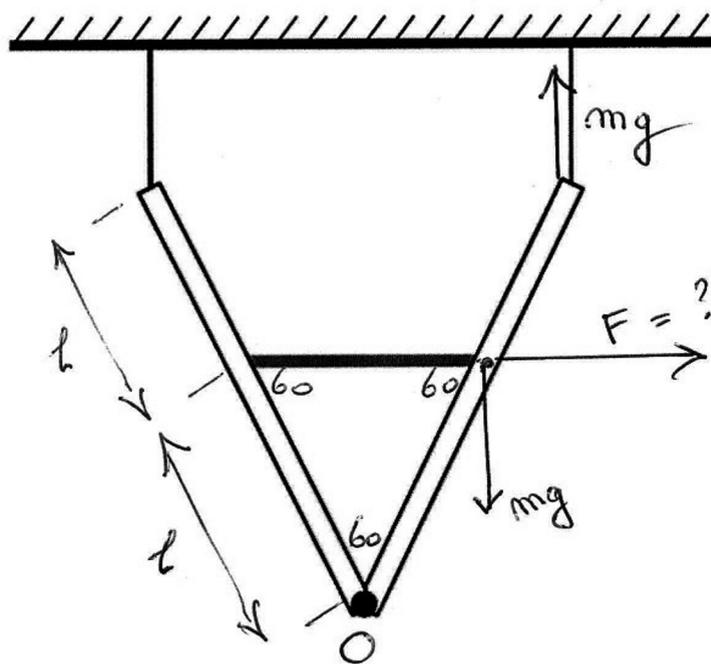
$$\omega = 2\pi f = \frac{3\pi}{L} \sqrt{\frac{T}{\mu}} \checkmark$$

$$v_{\text{max, transverse}} = \omega A = \frac{3\pi A}{L} \sqrt{\frac{T}{\mu}}$$

$$= \frac{3\pi \times 2 \times 10^{-3} \text{ m}}{2 \text{ m}} \sqrt{\frac{55 \text{ N}}{1.5 \times 10^{-3} / 2 \text{ kg/m}}}$$

$$= \boxed{2.55 \text{ m/s}}$$

# Mechanics A4



a) Zero torque w.r.t. point O:

$$mg \cdot \frac{1}{2}l + F \cdot \frac{1}{2}\sqrt{3}l - mg \cdot l = 0$$

$$F \cdot \frac{1}{2}\sqrt{3} = \frac{1}{2}mg$$

$$F = \frac{1}{3}\sqrt{3}mg = \frac{1}{3}\sqrt{3} \times 70 \times 9.8 = 396 \text{ N}$$

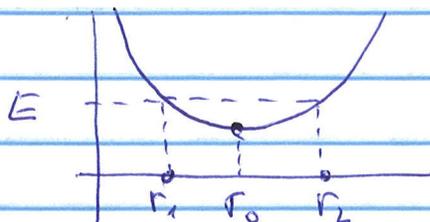
b) Because  $F > 0$  and  $F =$  force from crossbar on beam

we see that force from beam on crossbar points to the left  $\Rightarrow$  crossbar under compression.

# Mechanics B1

solutions *Olya F., mechanics, Aug 2015*

1. 
$$V_{\text{eff}}(r) = \frac{L^2}{2mr^2} + \frac{kr^2}{2}$$



(a) for the circular orbit  $r = r_0 = \text{const}$

$$\frac{dV_{\text{eff}}}{dr} = 0 \rightarrow -\frac{L^2}{mr_0^3} + kr_0 = 0$$

$$r_0^2 = \frac{L}{(mk)^{1/2}}, \quad E_0 = 2 \frac{L^2}{2m} \frac{(mk)^{1/2}}{L} = L \left(\frac{k}{m}\right)^{1/4}$$

(b) for noncircular orbit solve

$$E = \frac{L^2}{2mr^2} + \frac{kr^2}{2}$$

$$r_{1,2}^2 = \frac{E \mp \sqrt{E^2 - \frac{kL^2}{m}}}{k} \quad E \geq E_0$$

(c) expand  $V_{\text{eff}}$  near  $r = r_0$

$$\left. \frac{d^2V}{dr^2} \right|_{r=r_0} = \frac{3L^2}{mr_0^4} + k = \frac{3L^2}{m} \frac{mk}{L^2} + k = 4k$$

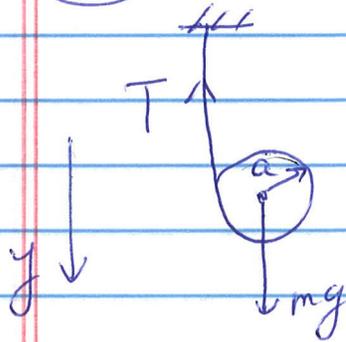
$$V_{\text{eff}} = V_{\text{eff}}(r_0) + \frac{1}{2} 4k (r - r_0)^2$$

This gives harmonic motion along radial coordinate with frequency  $\omega = \sqrt{\frac{4k}{m}}$

$$\text{period } T = \frac{2\pi}{\omega} = \pi \sqrt{\frac{m}{k}}$$

# Mechanics B2 solutions

2.



$$m\ddot{y} = mg - T \quad (1)$$

$$I\dot{\omega} = Ta$$

no slipping:  $\dot{\omega} = \frac{\dot{v}_y}{a} = \frac{\dot{y}}{a}$

$$I = \frac{2}{5}ma^2$$

$$\frac{2}{5}ma^2 \frac{\dot{v}_y}{a} = Ta$$

(2)  $T = \frac{2}{5}m\dot{v}_y$  substitute in (1)

$$m\dot{v}_y = mg - \frac{2}{5}m\dot{v}_y$$

(a)  $\dot{v}_y = \frac{5}{7}g$

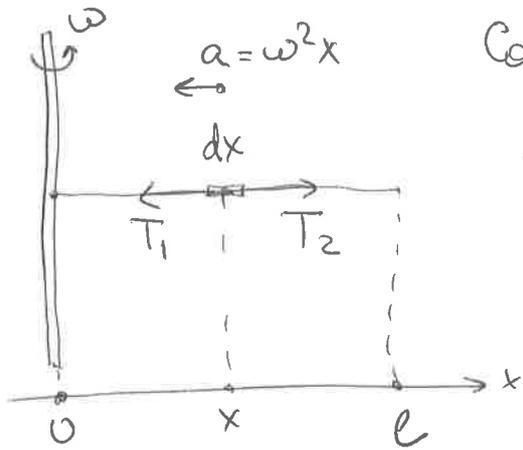
(b) From (2)  $T = \frac{2}{5}m \cdot \frac{5}{7}g = \frac{2}{7}mg$

Problem: oscillations (hard)

Solutions

Mechanics

## Mechanics B3 - solution



Consider a small segment of the string of the length  $dx$  at position  $x$ . Its mass is  $dm = \rho dx$ .

The second Newton's Law:

$$T_2 - T_1 = -dm \omega^2 x$$

$dT = -\rho dx \omega^2 x$ , integrate it:

$$T = -\frac{1}{2} \rho \omega^2 x^2 + C$$

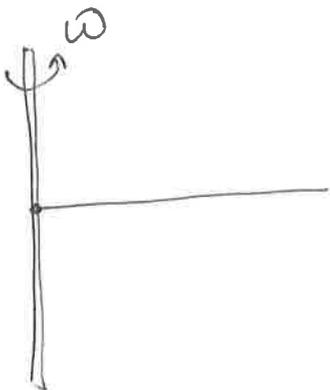
Use  $T(l) = 0$

$$T|_{x=l} = -\frac{1}{2} \rho \omega^2 l^2 + C = 0 \Rightarrow C = \frac{1}{2} \rho \omega^2 l^2$$

Finally:

$$T(x) = \frac{1}{2} \rho \omega^2 (l^2 - x^2)$$

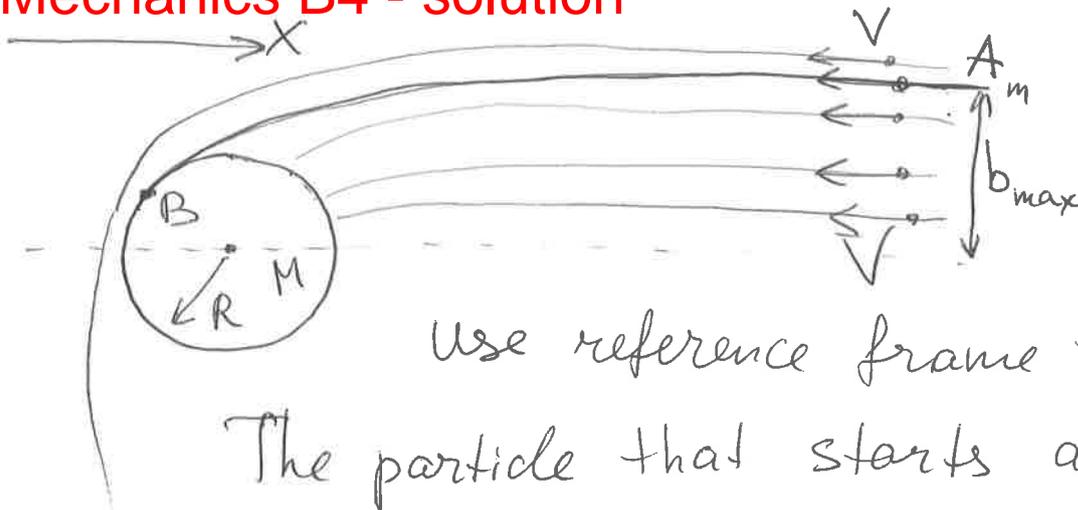
original drawing:



Problem: gravity (hard)  
Mechanics B4 - solution

Solutions

Mechanics



Use reference frame tied to the star.

The particle that starts at  $b = b_{\max}$  impact parameter has velocity  $-V$  at point A (far away).

At point B, this particle hits the surface of the star tangent to the surface. We can write angular momentum conservation: (with respect to the center of the star)

$$L_A = L_B$$

$$m V b_{\max} = m v_B R$$

↖ because velocity at B is tangent to surface

$$\Rightarrow v_B = \frac{V b_{\max}}{R}$$

We can also write the energy conservation:

$$K_A + U_A = K_B + U_B$$

$$\frac{mV^2}{2} + 0 = \frac{m v_B^2}{2} - \frac{GMm}{R} \Rightarrow V^2 = \left(\frac{V b_{\max}}{R}\right)^2 - \frac{2GM}{R}$$

$$b_{\max} = \frac{R}{V} \sqrt{V^2 + \frac{2GM}{R}}$$