UNL - Department of Physics and Astronomy

Preliminary Examination - Day 2
Friday, August 14, 2015

This test covers the topics of *Thermodynamics and Statistical Mechanics* (Topic 1) and *Mechanics* (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY

Thermodynamics and Statistical Mechanics Group A - Answer only two Group A questions

A1 A library box contains three kinds of books: 8 novels, 8 biographies, and 8 history books. What is the probability that two books picked at random from this box will be of different kinds?

A2 One mole of a monatomic ideal gas initially at temperature T_0 expands from volume V_0 to $2V_0$ at constant pressure. Calculate the work done by the gas and the heat absorbed by the gas.

A3 Using the first law of thermodynamics, write down the expression for the differential of internal energy dE in a reversible process for a system that can only do work by expansion, in variables V and S. Show how this leads to the definition of the Gibbs free energy G in variables T and P and write the expression for dG in these variables. Write down the Maxwell relation following directly from this last expression for dG.

A4 A reversible Carnot cycle is used in an electrically-powered heat pump to maintain a 20°C room temperature in a building while the outside air temperature is 4°C. Heat flow through the surface of the building at these conditions amounts to 10 kW. What is the minimum power that this heat pump must consume from the electric power source?

Thermodynamics and Statistical Mechanics Group B - Answer only two Group B questions

B1 A thermally conducting, uniform, and homogeneous bar of length *L*, cross section *A*, density ρ, and specific heat at constant pressure c_p is brought to a non-uniform temperature distribution by bringing one end in contact with a hot reservoir at a temperature T_{H} and the other end with a cold reservoir at a temperature T_{C}. Initially, the bar is in a steady state temperature distribution with a linear gradient. The bar is disconnected from the reservoirs, fully insulated immediately thereafter, and kept at constant pressure. Show that the change in entropy of the bar is

$$\Delta S = C_p\left(1+\ln\frac{T_f}{T_C}-\frac{T_H}{T_H-T_C}\ln\frac{T_H}{T_C}\right)$$

where $C_p = c_p\rho AL$, and $T_f = \left(T_{\mathrm{H}}+T_{\mathrm{C}}\right)/2$.

B2 Suppose that a one-dimensional unbiased random walker starts out at the origin $x=0$ at $t=0$ and takes unit length steps at regular intervals. The probability of a step to the right is *p*. Define a random variable, the *first-passage time*, equal to the number of steps *n* it will take for the walker to reach $x=+1$ for the *first* time . Find the probabilities for *n* equals 1, 2, …, 7.

B3 For a classical gas in equilibrium at temperature *T*, calculate the statistical averages $\langle v_x v_y\rangle$ and $\langle v_x^2 v_y^2\rangle$. Here v_x, v_y are the Cartesian components of the velocity of a given particle of mass m.

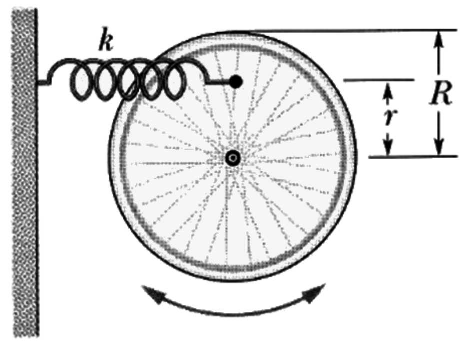
B4 A 3 kg bronze brick heated to 100°C is placed in a thermally insulated vessel with 1 kg of ice, which is initially at 0°C. The whole system is then allowed to reach equilibrium.

a. What is the final temperature of the system?
b. How much ice has thawed?
c. What is the total change of entropy of the system?

Reference data: Specific heat of water: 4186 J/(kg·K). Specific heat of ice: 2050 J/(kg·K). Specific heat of bronze: 435 J/(kg·K). It takes 334 joules to melt 1 gram of ice.

Mechanics Group A - Answer only two Group A questions

A1 A wheel with massless spokes is free to rotate about its fixed axle. A spring is attached to one of its spokes a distance r from the axle, as shown in the figure.



a. Assuming that the wheel is a hoop of mass m and radius R, what is the angular frequency ω of small oscillations of this system in terms of m, R, r, and the spring constant k?

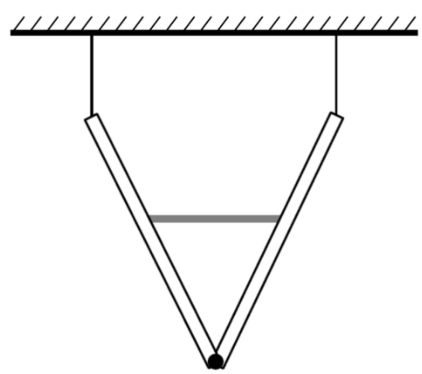
What is ω if

b. $r = R$, and
c. $r = 0$?

A2 A projectile is fired vertically from Earth's surface with an initial speed of 10 km/s. Neglecting air drag, how far above the surface of Earth will it go?

A3 A uniform string of mass 1.5 grams is put under a tension of 55 N between two pegs 2 m apart. The string is vibrating in the second harmonic above its fundamental frequency, with an amplitude of 2 mm. For this motion, what is the highest speed any part of the string attains?

A4 Two uniform beams of mass 70 kg each are suspended from the ceiling as shown by cables and are connected at their bottom ends by a frictionless hinge. A massless horizontal crossbar connects their midpoints as shown. The angle between the beams is 60°.



a. What is the magnitude of the force exerted by the crossbar on the right-hand beam?
b. Is the crossbar under compression or tension?

Mechanics Group B - Answer only two Group B questions

B1 A particle of mass m is moving in the spherically-symmetric potential

$$V(r) = \tfrac{1}{2}kr^2$$

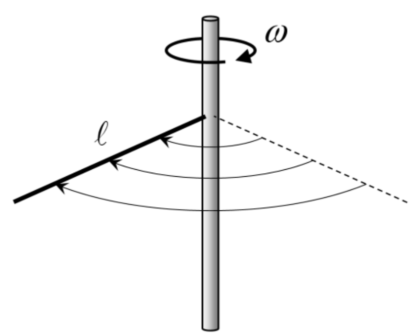
with orbital angular momentum L.

a. Suppose its orbit is circular. Find the radius of the orbit and the corresponding total energy.
b. For a noncircular orbit and given energy E, find the smallest and largest distances of the particle from the center.
c. Suppose the orbit is close to circular. What is the period of the particle's motion in the radial direction?

B2 A uniform solid ball has a few turns of light string wound around it. The end of the string is held steady and the ball is allowed to fall under gravity, while the string remains vertical.

a. What is the acceleration of the center of the ball?
b. What is the tension in the string, if the ball's mass is m?

B3 Suppose you have a string of a uniform mass per unit length ρ and length ℓ attached at one end to a rotating vertical pole and free at the other end. The string rotates in a horizontal plane at an angular frequency ω (neglect gravity) as shown in the figure. Find the tension in the string as a function of the distance to the pole.



B4 A heavy star of mass M and radius R moves with velocity $\mathbf{v}$ through a very dilute gas of mass density ρ. It pulls particles toward itself by its gravitational field and captures all of the particles that strike its surface. Find the maximum impact parameter that a particle can have to still be pulled in and absorbed by the star. (Impact parameter is the distance of a far-away particle from the line going through the center of the star and parallel to the star's velocity). You may assume that thermal speeds of the particles are negligible relative to $v = |\mathbf{v}|$ and that the interactions of particles with each other can be neglected.

Physical Constants

speed of light .... $c = 2.998 \times 10^{8}$ m/s

Planck's constant .... $h = 6.626 \times 10^{-34}$ J·s

Planck's constant / 2π.... $\hbar = 1.055 \times 10^{-34}$ J·s

Boltzmann constant ...... $k_B = 1.381 \times 10^{-23}$ J/K

elementary charge .... $e = 1.602 \times 10^{-19}$ C

electric permittivity ...... $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m

magnetic permeability ... $\mu_0 = 1.257 \times 10^{-6}$ H/m

molar gas constant.......... $R = 8.314$ J / mol·K

Avogrado constant .... $N_A = 6.022 \times 10^{23}$ mol^{-1}

electrostatic constant ... $k = (4\pi\varepsilon_0)^{-1} = 8.988 \times 10^{9}$ m/F

electron mass .... $m_{el} = 9.109 \times 10^{-31}$ kg

electron rest energy...... 511.0 keV

Compton wavelength .. $\lambda_C = h / m_{el} c = 2.426$ pm

proton mass .... $m_p = 1.673 \times 10^{-27}\,\text{kg} = 1836 m_{el}$

1 bohr .... $a_0 = \hbar^2 / k e^2 m_{el} = 0.5292$ Å

1 hartree (= 2 rydberg) ... $E_h = \hbar^2 / m_{el} a_0^{\,2} = 27.21$ eV

gravitational constant ... $G = 6.674 \times 10^{-11}$ m^3 / kg s^2

hc .... $hc = 1240$ eV·nm

Equations That May Be Helpful

MECHANICS

Properties of the Earth:

- Mass: $M = 5.98 \times 10^{24}$ kg
- Radius: $R = 6.38 \times 10^{6}$ m
- Gravitational acceleration at surface: $g = 9.81$ m/s^2

Gauss's Law for gravity

$$\oint_S \mathbf{g} \cdot d\mathbf{A} = -4\pi G m_{\text{encl.}}$$

THERMODYNAMICS

Clausius' theorem: $\sum_{i=1}^{N} \frac{Q_i}{T_i} \le 0$, which becomes $\sum_{i=1}^{N} \frac{Q_i}{T_i} = 0$ for a reversible cyclic process of N steps.

For adiabatic processes in an ideal gas with constant heat capacity, $pV^{\gamma} = \text{const.}$

$$\frac{dp}{dT} = \frac{\lambda}{T \Delta V}$$

$$C_V = \left(\frac{\delta Q}{dT}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V \qquad C_p = \left(\frac{\delta Q}{dT}\right)_p = T\left(\frac{\partial S}{\partial T}\right)_p \qquad TdS = C_V dT + T\left(\frac{\partial S}{\partial V}\right)_T dV$$

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence : $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

Curl : $\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$

Laplacian : $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$

Divergence : $\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$

Curl :

$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence : $\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$

Curl : $\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$

Laplacian : $\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint \mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint \mathbf{A}\cdot d\mathbf{l}$

INTEGRALS

$f(x)$		$\int_0^\infty f(x)\,dx$
e^{-ax^2}	....	$\frac{\sqrt{\pi}}{2\sqrt{a}}$
xe^{-ax^2}	....	$\frac{1}{2a}$
$x^2e^{-ax^2}$	....	$\frac{\sqrt{\pi}}{4a^{3/2}}$
$x^3e^{-ax^2}$	....	$\frac{1}{2a^2}$
$x^4e^{-ax^2}$	....	$\frac{3\sqrt{\pi}}{8a^{5/2}}$
$x^5e^{-ax^2}$	....	$\frac{1}{a^3}$
$x^6e^{-ax^2}$	....	$\frac{15\sqrt{\pi}}{16a^{7/2}}$

$$\int_0^\infty \frac{1}{1+ay^2}dy = \pi / 2a^{1/2}$$

$$\int_0^\infty y^n e^{-y/a}dy = n!a^{n+1}$$

$$\begin{aligned}
\int x^{-2}\ln(x)\,dx &= -\frac{\ln(x)}{x} - \frac{1}{x} \\
\int x^{-1}\ln(x)\,dx &= \tfrac{1}{2}\ln^2(x) \\
\int \ln(x)\,dx &= x\ln(x) - x \\
\int x\ln(x)\,dx &= \tfrac{1}{2}x^2\ln(x) - \tfrac{1}{4}x^2 \\
\int x^2\ln(x)\,dx &= \tfrac{1}{3}x^3\ln(x) - \tfrac{1}{9}x^3
\end{aligned}$$

$$\int \ln(1+x)dx = (x+1)\big(\ln(x+1)-1\big)+C$$

$$\begin{aligned}
\int \frac{r^3dr}{(x^2+r^2)^{3/2}} &= (r^2+x^2)^{1/2} + \frac{x^2}{(r^2+x^2)^{1/2}} \\
\int \frac{dx}{a^2+x^2} &= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) \qquad \left|\tan^{-1}\left(\frac{x}{a}\right)\right| < \frac{\pi}{2} \\
\int \frac{xdx}{a^2+x^2} &= \frac{1}{2}\ln\left(a^2+x^2\right) \\
\int \frac{dx}{x(a^2+x^2)} &= \frac{1}{2a^2}\ln\left(\frac{x^2}{a^2+x^2}\right) \\
\int \frac{dx}{a^2x^2-b^2} &= \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right) \\
&= -\frac{1}{ab}\coth^{-1}\left(\frac{ax}{b}\right) \\
&= -\frac{1}{ab}\tanh^{-1}\left(\frac{ax}{b}\right), \quad a^2x^2 < b^2
\end{aligned}$$