UNL - Department of Physics and Astronomy

Preliminary Examination - Day 1
Thursday, August 11, 2016

This test covers the topics of *Quantum Mechanics* (Topic 1) and *Electrodynamics* (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

Note: If you do more than two problems in a group, only the first two (in the order they appear in this test) will be graded. For instance, if you do problems A1, A3, and A4, only A1 and A3 will be graded.

WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY

<u>**Quantum Mechanics Group A** - Answer only two Group A questions</u>

A1 A light source emits, isotropically, 40 W of radiation with wavelength 500 nm. What is the photon flux (photons/cm² s) at a distance 4 km from the source?

A2 Consider a free particle moving in one dimension with mass m and energy E. Write down all possible solutions of the time-dependent Schrödinger equation which are (*a*) eigenstates of momentum; (*b*) eigenstates of parity. Are the answers to parts (*a*) and (*b*) identical? Explain why or why not.

A3 The linear operator G has the property $GG + 2E = 3G$, where E is the identity operator. What eigenvalue(s) could G have, if any?

A4 The wave function of a particle moving in one dimension is given by

$$\psi(x) = \begin{cases} C & \text{for } -b \le x \le 2b \\ 0 & \text{elsewhere} \end{cases}$$

where $b > 0$ and C is a constant. Calculate the expectation value of the parity operator.

<u>Quantum Mechanics Group B</u> <u>- Answer only two Group B questions</u>

B1 A particle moving on a sphere is described by the wavefunction (in spherical coordinates)

$$\psi = C \sin\theta \sin\phi ,$$

where C is a constant.

a. What is the expectation value of L_z for this state?

b. What are the possible outcomes of the measurements of L_z, and what are their probabilities?

c. What is the expectation value of L_x ?

d. What is the expectation value of L^2 ?

e. Find the normalization constant *C*, and the expectation values of $\cos\theta$ and $\cos^2\theta$.

B2 A particle is placed in an infinitely deep, one-dimensional box with the ends at $x = 0$ and $x = a$. At $t = 0$ the particle's wave function is given by

$$\psi(x) = \delta(x - x_0) , \quad \text{with } 0 < x_0 < a ,$$

where $\delta(x - x_0)$ is the Dirac delta function.

a. At $t = t_0$ the particle's energy is measured. What are the possible outcomes of this measurement, and what are the corresponding probabilities?

b. Suppose the measurement in part (*a*) gives $E = E_1$, the energy of the ground state of this particle-in-a-box problem. What is the probability to find the particle between $x = a/4$ and $x = 3a/4$ at $t > t_0$?

B3 Consider a harmonic oscillator with Hamiltonian $\hat{H} = \hat{p}^2 / 2m + \frac{1}{2} m\omega_0{}^2 \hat{x}^2 = (\hat{a}^\dagger \hat{a} + \frac{1}{2})\hbar\omega_0$, and stationary states given by $\hat{H} | n\rangle = E_n | n\rangle = (n + \frac{1}{2})\hbar\omega_0 | n\rangle$. The annihilation operator $\hat{a}$ is defined by $\hat{a} = \frac{1}{2}\sqrt{2}\,\beta\left(\hat{x} + \dfrac{i}{m\omega_0}\hat{p}\right)$ (with $\beta^2 = m\omega_0 / \hbar$). We have $| n\rangle = \dfrac{(a^\dagger)^n}{\sqrt{n!}} | 0\rangle$.

a. Show that $\hat{a}^\dagger | n\rangle = \sqrt{n+1}\, | n+1\rangle$.

It can also be shown that $\hat{a} | n\rangle = \sqrt{n}\, | n-1\rangle$. You don't have to prove this.

b. Show that $\langle x\rangle = \langle n | x | n\rangle = 0$.

c. Calculate $\langle n | x^2 | n\rangle = \langle x^2\rangle$. *Hint: Write x^2 in terms of a and $a^\dagger$.*

d. Calculate Δx.

e. Show that $\langle p\rangle = \langle n | p | n\rangle = 0$.

f. Calculate $\langle n | p^2 | n\rangle = \langle p^2\rangle$. *Hint: Write p^2 in terms of a and $a^\dagger$.*

g. Calculate Δp.

h. Verify that Heisenberg's uncertainty relation is satisfied. In which case is the uncertainty smallest?

B4 We consider the parity operator, which we will denote by $\hat{\Pi}$.

a. Show that the eigenvalues of $\hat{\Pi}$ are $\Pi = \pm 1$. *Hint: use $\hat{\Pi}\hat{\Pi} = \hat{I}$, where $\hat{I}$ is the identity operator.*

b. Show that $\langle\Pi\rangle$, the expectation value of the parity, is limited to the range $-1 \le \langle\Pi\rangle \le 1$.

We now consider the harmonic oscillator, with Hamiltonian

$$\hat{H} = \hat{p}\hat{p} / 2m + \tfrac{1}{2} m\omega_0{}^2 \hat{x}\hat{x} \ ,$$

where $\hat{x}$ and $\hat{p}$ are the position and momentum operator, respectively. The stationary states are $| n\rangle$, where $\hat{H} | n\rangle = (n + \frac{1}{2})\hbar\omega_0 | n\rangle$ ($n = 0, 1, 2, \ldots$).

c. Show that $[\hat{H}, \hat{\Pi}] = \hat{0}$.

At time $t = 0$, the particle's state is given by $| \psi\rangle(t = 0) = \frac{1}{2}\sqrt{3}\, | 0\rangle - c\, | 3\rangle$, where c is a positive real constant.

d. Find the value of c that normalizes $| \psi\rangle(t = 0)$.

e. Calculate $\langle\Pi\rangle$ for $t = 0$.

f. Calculate $\langle\Pi\rangle(t)$ for $t > 0$.

Electrodynamics Group A - Answer only two Group A questions

A1 Charge density is distributed on a solid sphere of radius 5 cm in such a way that the volume charge density, ρ, is spherically symmetric and equals $50r^2\ \text{mC/m}^5$. What is the electrical potential of the center of the sphere relative to that at $r=\infty$?

A2 A point charge of $-15\ \mu\text{C}$ sits at the origin of a Cartesian coordinate system. What is the absolute value of the electric flux integral over the surface $z = 1\ \text{cm}$, $x^2 + y^2 \le 1\ \text{cm}^2$?

A3 In the adjacent Fig. 1(a), a point charge q_1 and a point charge q_2 are fixed in place on the x axis, 8 cm apart. A third point charge $q_3 = 8\times10^{-19}$ C is placed on the x axis between q_1 and q_2. Figure 1(b) gives the x component of the net force on q_3 as a function of the position x at which the particle q_3 is placed.

Fig. 1(a)

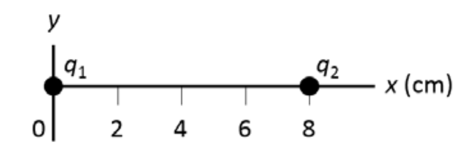
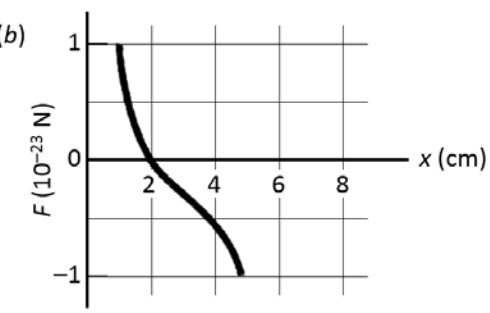


Fig. 1(b)



a. What is the sign of the charge on particle q_1?

b. What is the ratio q_2 / q_1?

A4 Figure 2(a) shows a length of wire bent into a circular coil of one turn. In Fig. 2(b) an identical length of the same kind of wire has been bent to make a coil of two turns. Each coil carries the same current I. The two coils are very far apart.

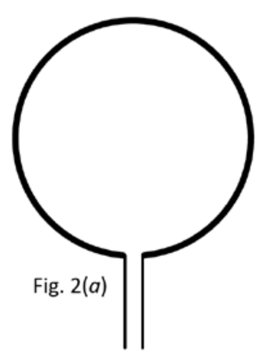


Fig. 2(a)

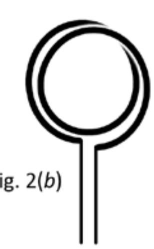


Fig. 2(b)

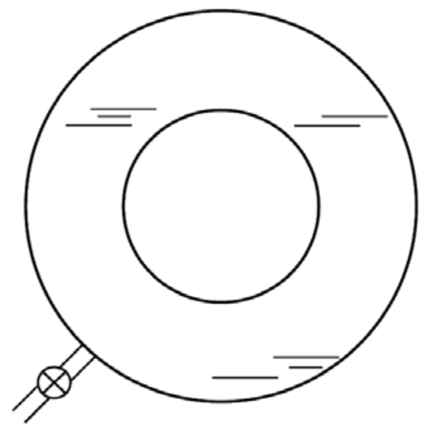
a. If B_a and B_b are the magnitudes of the magnetic fields at the centers of the two coils, what is the ratio B_b / B_a?

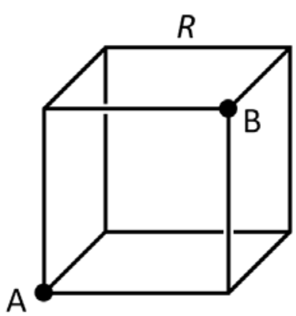
b. What is the ratio μ_b / μ_a of the dipole moment magnitudes of the coils?

Electrodynamics Group B - Answer only two Group B questions

B1 A long cylindrical insulator of radius R is placed parallel to, and a distance $d \gg R$ above an infinite, flat, silver plate. The rod has a linear charge density λ comprising uniformly-distributed charge throughout its volume. What is the electrostatic potential difference between the surface of the rod and the plate? You may neglect end effects and the non-ideal nature of the conductor.

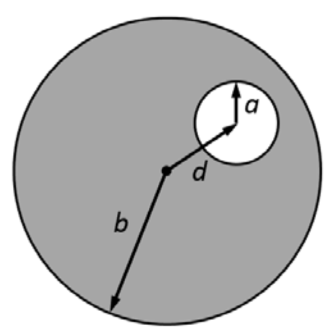
B2 Consider a spherical capacitor consisting of an inner conducting shell of radius 1 cm and an outer conducting shell of radius 2 cm. The space between the shells is filled with oil of dielectric constant = 2. On the inner and outer shells are placed, uniformly, charges of +5 μC and –5 μC, respectively. How much electrostatic work (as opposed to gravitational or mechanical work) is required to pump the oil out of the capacitor? Explain why your answer has the sign it does. Would the sign change if the initial charge on the two spheres was reversed? Why or why not?





B3 The 12 edges of a cube consist of resistors of equal resistance R, which are joined at the corners. What is the effective resistance between two opposite corners of a face of the cube (A and B)?

B4 An eccentric hole of radius a is bored parallel to the axis of a straight circular cylinder of radius b ($b > a$).The two axes are at a distance d apart. A current I flows in the cylinder, with uniform current density parallel to its symmetry axis. What is the magnetic field at the center of the hole?



Physical Constants

speed of light .... $c = 2.998 \times 10^{8}$ m/s

Planck's constant .... $h = 6.626 \times 10^{-34}$ J·s

Planck's constant / 2π.... $\hbar = 1.055 \times 10^{-34}$ J·s

Boltzmann constant ...... $k_{\mathrm{B}} = 1.381 \times 10^{-23}$ J/K

elementary charge .... $e = 1.602 \times 10^{-19}$ C

electric permittivity ...... $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m

magnetic permeability ... $\mu_0 = 1.257 \times 10^{-6}$ H/m

molar gas constant.......... $R = 8.314$ J / mol·K

Avogrado constant .... $N_{\mathrm{A}} = 6.022 \times 10^{23}$ mol^{-1}

electrostatic constant ... $k = (4\pi\varepsilon_0)^{-1} = 8.988 \times 10^{9}$ m/F

electron mass .... $m_{\mathrm{el}} = 9.109 \times 10^{-31}$ kg

electron rest energy...... 511.0 keV

Compton wavelength .. $\lambda_{\mathrm{C}} = h / m_{\mathrm{el}} c = 2.426$ pm

proton mass .... $m_{\mathrm{p}} = 1.673 \times 10^{-27}\,\mathrm{kg} = 1836 m_{\mathrm{el}}$

1 bohr............................ $a_0 = \hbar^2 / k e^2 m_{\mathrm{el}} = 0.5292$ Å

1 hartree (= 2 rydberg) ... $E_{\mathrm{h}} = \hbar^2 / m_{\mathrm{el}} a_0{}^2 = 27.21$ eV

gravitational constant ... $G = 6.674 \times 10^{-11}$ m^3 / kg s^2

hc .... $hc = 1240$ eV·nm

Equations That May Be Helpful

TRIGONOMETRY

$$\sin\alpha\sin\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$
$$\cos\alpha\cos\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$
$$\sin\alpha\cos\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$
$$\cos\alpha\sin\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$$

QUANTUM MECHANICS

Particle in one-dimensional, infinitely-deep box with walls at $x = 0$ and $x = a$:

Stationary states $\psi_n = (2/a)^{1/2} \sin(n\pi x / a)$, energy levels $E_n = n^2 \dfrac{\pi^2\hbar^2}{2ma^2}$

Angular momentum: $[L_x, L_y] = i\hbar L_z$ *et cycl.*

Ladder operators:
$$L_{\pm} = L_x \pm i L_y$$
$$L_{+} \,|\, \ell, m\rangle = \hbar\sqrt{(\ell+m+1)(\ell-m)}\,|\, \ell, m+1\rangle$$
$$L_{-} \,|\, \ell, m\rangle = \hbar\sqrt{(\ell+m)(\ell-m+1)}\,|\, \ell, m-1\rangle$$

Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;,\; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \;,\; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Spherical harmonics $Y_\ell^m(\theta,\phi)$

$$Y_0^0(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{1}{\pi}}$$

$$\begin{aligned}
Y_1^{-1}(\theta,\varphi) &= \frac{1}{2}\sqrt{\frac{3}{2\pi}} \cdot e^{-i\varphi} \cdot \sin\theta &&= \frac{1}{2}\sqrt{\frac{3}{2\pi}} \cdot \frac{(x-iy)}{r} \\
Y_1^{0}(\theta,\varphi) &= \frac{1}{2}\sqrt{\frac{3}{\pi}} \cdot \cos\theta &&= \frac{1}{2}\sqrt{\frac{3}{\pi}} \cdot \frac{z}{r} \\
Y_1^{1}(\theta,\varphi) &= -\frac{1}{2}\sqrt{\frac{3}{2\pi}} \cdot e^{i\varphi} \cdot \sin\theta &&= -\frac{1}{2}\sqrt{\frac{3}{2\pi}} \cdot \frac{(x+iy)}{r}
\end{aligned}$$

$$\begin{aligned}
Y_2^{-2}(\theta,\varphi) &= \frac{1}{4}\sqrt{\frac{15}{2\pi}} \cdot e^{-2i\varphi} \cdot \sin^2\theta &&= \frac{1}{4}\sqrt{\frac{15}{2\pi}} \cdot \frac{(x-iy)^2}{r^2} \\
Y_2^{-1}(\theta,\varphi) &= \frac{1}{2}\sqrt{\frac{15}{2\pi}} \cdot e^{-i\varphi} \cdot \sin\theta \cdot \cos\theta &&= \frac{1}{2}\sqrt{\frac{15}{2\pi}} \cdot \frac{(x-iy)z}{r^2} \\
Y_2^{0}(\theta,\varphi) &= \frac{1}{4}\sqrt{\frac{5}{\pi}} \cdot (3\cos^2\theta - 1) &&= \frac{1}{4}\sqrt{\frac{5}{\pi}} \cdot \frac{(2z^2 - x^2 - y^2)}{r^2} \\
Y_2^{1}(\theta,\varphi) &= \frac{-1}{2}\sqrt{\frac{15}{2\pi}} \cdot e^{i\varphi} \cdot \sin\theta \cdot \cos\theta &&= \frac{-1}{2}\sqrt{\frac{15}{2\pi}} \cdot \frac{(x+iy)z}{r^2} \\
Y_2^{2}(\theta,\varphi) &= \frac{1}{4}\sqrt{\frac{15}{2\pi}} \cdot e^{2i\varphi} \cdot \sin^2\theta &&= \frac{1}{4}\sqrt{\frac{15}{2\pi}} \cdot \frac{(x+iy)^2}{r^2}
\end{aligned}$$

ELECTROSTATICS

$\oiint_S \mathbf{E}\cdot d\mathbf{A} = \dfrac{q_{\text{enc}}}{\varepsilon_0}$ $\qquad$ $\mathbf{E} = -\nabla V$ $\qquad$ $\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{E}\cdot d\boldsymbol{\ell} = V(\mathbf{r}_1) - V(\mathbf{r}_2)$ $\qquad$ $V(\mathbf{r}) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$

Work done $W = -\int_{\mathbf{a}}^{\mathbf{b}} q\mathbf{E}\cdot d\boldsymbol{\ell} = q\left[V(\mathbf{b}) - V(\mathbf{a})\right]$ Energy stored in elec. field: $W = \frac{1}{2}\varepsilon_0 \int_V E^2 d\tau = Q^2/2C$

MAGNETOSTATICS

Lorentz force: $\mathbf{F} = q\mathbf{E} + q(\mathbf{v}\times\mathbf{B})$ Current densities: $I = \int \mathbf{J}\cdot d\mathbf{A}$, $I = \int \mathbf{K}\cdot d\boldsymbol{\ell}$

Biot-Savart law: $\mathbf{B}(\mathbf{r}) = \dfrac{\mu_0}{4\pi}\int \dfrac{I d\boldsymbol{\ell}\times\hat{\mathbf{R}}}{R^2}$ ($\mathbf{R}$ is vector from source point to field point $\mathbf{r}$)

For straight wire segment: $B = \dfrac{\mu_0 I}{4\pi s}\left[\sin\theta_2 - \sin\theta_1\right]$ where s is the perpendicular distance from wire.

Infinitely long solenoid: B-field inside is $B = \mu_0 n I$ (n is number of turns per unit length)

Ampere's law: $\oint \mathbf{B}\cdot d\boldsymbol{\ell} = \mu_0 I_{\text{enclosed}}$

Magnetic dipole moment of a current distribution is given by $\mathbf{m} = I\int d\mathbf{a}$.

Force on magnetic dipole: $\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$

Torque on magnetic dipole: $\boldsymbol{\tau} = \mathbf{m}\times\mathbf{B}$

B-field of magnetic dipole: $\mathbf{B}(\mathbf{r}) = \dfrac{\mu_0}{4\pi}\dfrac{1}{r^3}\left[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\right]$

The dipole-dipole interaction energy is $U_{\text{DD}} = \dfrac{\mu_0}{4\pi R^3}\left[(\mathbf{m}_1\cdot\mathbf{m}_2) - 3(\mathbf{m}_1\cdot\hat{\mathbf{R}})(\mathbf{m}_2\cdot\hat{\mathbf{R}})\right]$, where $\mathbf{R} = \mathbf{r}_1 - \mathbf{r}_2$.

Maxwell's Equations in vacuum

1. $\nabla \cdot \mathbf{E} = \dfrac{\rho}{\varepsilon_0}$ Gauss' Law
2. $\nabla \cdot \mathbf{B} = 0$ no magnetic charge
3. $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ Faraday's Law
4. $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \varepsilon_0 \mu_0 \dfrac{\partial \mathbf{E}}{\partial t}$ Ampere's Law with Maxwell's correction

Maxwell's Equations in linear, isotropic, and homogeneous (LIH) media

1. $\nabla \cdot \mathbf{D} = \rho_{\mathrm{f}}$ Gauss' Law
2. $\nabla \cdot \mathbf{B} = 0$ no magnetic charge
3. $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ Faraday's Law
4. $\nabla \times \mathbf{H} = \mathbf{J}_{\mathrm{f}} + \dfrac{\partial \mathbf{D}}{\partial t}$ Ampere's Law with Maxwell's correction

Induction

Alternative way of writing Faraday's Law: $\oint \mathbf{E} \cdot d\boldsymbol{\ell} = -\dfrac{d\Phi}{dt}$

Mutual and self inductance: $\Phi_2 = M_{21} I_1$, and $M_{21} = M_{12}$; $\Phi = LI$

Energy stored in magnetic field: $W = \frac{1}{2}\mu_0^{-1} \int_V B^2 d\tau = \frac{1}{2} LI^2 = \frac{1}{2} \oint \mathbf{A} \cdot \mathbf{I}\, d\ell$

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl : $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

INTEGRALS

$f(x)$	$\int_0^\infty f(x)\,dx$
e^{-ax^2}	$\dfrac{\sqrt{\pi}}{2\sqrt{a}}$
xe^{-ax^2}	$\dfrac{1}{2a}$
$x^2e^{-ax^2}$	$\dfrac{\sqrt{\pi}}{4a^{3/2}}$
$x^3e^{-ax^2}$	$\dfrac{1}{2a^2}$
$x^4e^{-ax^2}$	$\dfrac{3\sqrt{\pi}}{8a^{5/2}}$
$x^5e^{-ax^2}$	$\dfrac{1}{a^3}$
$x^6e^{-ax^2}$	$\dfrac{15\sqrt{\pi}}{16a^{7/2}}$

$$\int_0^\infty \frac{1}{1+ay^2}dy = \pi / 2a^{1/2}$$

$$\int_0^\infty y^n e^{-ay} dy = \frac{n!}{a^{n+1}}$$

$$\int \arcsin x\,dx = x\arcsin x + \sqrt{1-x^2}$$

$$\int \arccos x\,dx = x\arccos x - \sqrt{1-x^2}$$

$$\int \arctan x\,dx = x\arctan x - \tfrac{1}{2}\ln(x^2+1)$$

$$\int \operatorname{arsinh} x\,dx = x\operatorname{arsinh} x - \sqrt{x^2+1}$$

$$\int \operatorname{arcosh} x\,dx = x\operatorname{arcosh} x - \sqrt{x-1}\sqrt{x+1}$$

$$\int \operatorname{artanh} x\,dx = x\operatorname{artanh} x + \tfrac{1}{2}\ln(1-x^2)$$

$$\int \frac{r^3 dr}{(x^2+r^2)^{3/2}} = (r^2+x^2)^{1/2} + \frac{x^2}{(r^2+x^2)^{1/2}}$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) \qquad \left|\tan^{-1}\left(\frac{x}{a}\right)\right| < \frac{\pi}{2}$$

$$\int \frac{xdx}{a^2+x^2} = \frac{1}{2}\ln\left(a^2+x^2\right)$$

$$\int \frac{dx}{x(a^2+x^2)} = \frac{1}{2a^2}\ln\left(\frac{x^2}{a^2+x^2}\right)$$

$$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right)$$

$$= -\frac{1}{ab}\coth^{-1}\left(\frac{ax}{b}\right)$$

$$= -\frac{1}{ab}\tanh^{-1}\left(\frac{ax}{b}\right), \quad a^2x^2 < b^2$$