UNL - Department of Physics and Astronomy

Preliminary Examination - Day 2
Friday, August 12, 2016

This test covers the topics of *Thermodynamics and Statistical Mechanics* (Topic 1) and *Mechanics* (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

Note: If you do more than two problems in a group, only the first two (in the order they appear in this test) will be graded. For instance, if you do problems A1, A3, and A4, only A1 and A3 will be graded.

WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY

Thermodynamics and Statistical Mechanics Group A - Answer only two Group A questions

A1 The radioactive nucleus carbon-10 (^{10}C) has a half-life of 20 s. Suppose you have 13 such nuclei. What is the probability P that, after 30 s, exactly 4 of them (no more, no less) haven't decayed yet?

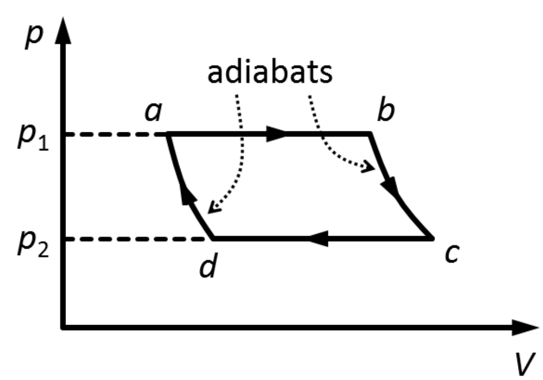
A2 Work is done on an ideal gas at constant temperature T. The volume of the gas changes from V_1 to V_2. How much energy does the gas give or take from its surrounding?

A3 Derive the Maxwell relation

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial P}{\partial S}\right)_V .$$

Hint: you can find a list of the full differentials for thermodynamic potentials in the formula sheet.

A4 Consider an engine working in a reversible cycle and using an ideal gas with constant heat capacity c_p as the working substance. The cycle consists of two processes at constant pressure, joined by two adiabats. Find the efficiency of this engine in terms of p_1 and p_2 as well as γ for the adiabatic process.



Thermodynamics and Statistical Mechanics Group B - Answer only two Group B questions

B1 *Hint for this problem: use the triple product.*

Consider a gas where the internal energy U and pressure p are given by

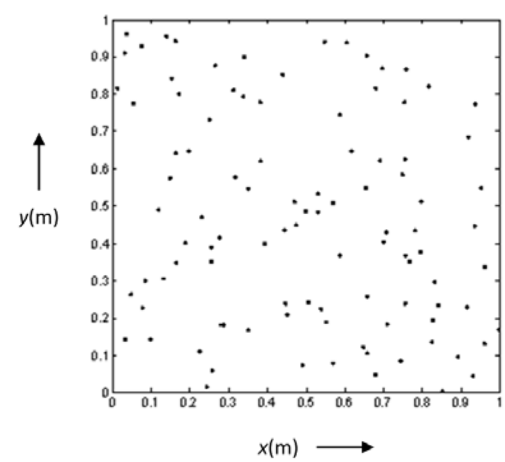
$$U = \frac{f}{2} N k_B T - b\frac{N^2}{V} \quad \text{and}$$

$$p = \frac{N k_B T}{V} - b\frac{N^2}{V^2}.$$

Here, N is the number of molecules, V is the volume, f is the number of degrees of freedom, and b is a constant.

a. Find an expression for the entropy as a function of T and V.
b. Derive an expression for the connection between T and V for an adiabatic process.
c. Find expressions for c_V, the specific heat per particle at constant volume, and c_p, the specific heat per particle at constant pressure.

B2 Rain falls vertically down on a horizontal square surface of the size 1 m × 1 m. Consider the distribution of the rain drops to be uniform. The position of each rain drop can be described by its (x,y) coordinates, as shown in the figure. Find the average value of the minimum of the x and y coordinates for the rain drops, i.e., $\langle \min(x,y) \rangle$.



B3 Two fluids, F_1 and F_2, of fixed volumes and constant heat capacities C_1 and C_2, are initially at temperatures T_1 and T_2 (with $T_1 > T_2$), respectively. They are adiabatically insulated from each other. A quasistatically acting Carnot engine uses F_1 as a heat source and F_2 as a heat sink, and acts between the systems until they reach a common temperature T_0. Obtain the expression for T_0 and for the work done by the Carnot engine.

B4 The thermodynamics of a classical paramagnetic system is expressed by the following variables: magnetization M, magnetic field B, and absolute temperature T. The equation of state is

$$M = CB/T, \quad \text{where } C \text{ is the Curie constant.}$$

The system's internal energy is

$$U = -MB.$$

The increment of work done by the system upon the external environment is

$$dW = MdB.$$

a. Write an expression for the heat input, dQ, to the system in terms of thermodynamic variables M and B:

$$dQ = (\quad)dM + (\quad)dB.$$

b. Find the expression for the differential of the system entropy:

$$dS = (\quad)dM + (\quad)dB.$$

c. Finally, derive an expression for the entropy S.

Mechanics Group A - Answer only two Group A questions

A1 Two blocks with different masses are attached to either end of a light rope that passes over a light, frictionless pulley suspended from the ceiling. The masses are released from rest, and the more massive one starts to descend. After this block has descended 1.2 m, its speed is 3.0 m/s. If the total mass of the two blocks is 22 kg, what is the mass of each block?

A2 For an object that rolls without slipping a certain fraction of the total kinetic energy is rotational. For which of the following three objects is this fraction greatest: (I) a uniform solid cylinder; (II) a uniform solid sphere; or (III) a thin-walled, hollow sphere?

A3 An ant clings to the tip of a helicopter blade of length 3 m, which rotates in a horizontal plane about one of its ends. The blade starts from rest at $t = 0$ and, under uniform angular acceleration, attains an angular velocity of 100 revolutions per minute after 10 revolutions. What is the magnitude of the ant's acceleration vector at $t = 1.35$ s?

A4 A shallow reservoir at an elevation above sea level of 340 m contains 27 acre-feet of water. If a pump situated at sea level is expected to take in water from the ocean and fill this reservoir in 1 day, what minimum power must be supplied to its motor, assuming no transfer loss of the water and 100% motor efficiency? Note: $1 \text{ acre-foot} = 1233 \text{ m}^3$.

Mechanics Group B - Answer only two Group B questions

B1 A mass m moves in a circular orbit of radius R_0 under the influence of a central force whose potential is $-km/r^n$. Show that the circular orbit is stable under small oscillations (that is, the mass will oscillate about the circular orbit) if $n<2$.

B2 A planet has the same average density as the Earth, 5.51 g/cm^3. All bodies at rest on the surface of the planet at its equator are weightless.

a. What is the period of revolution of the planet about its axis?

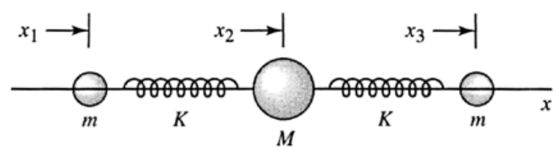
b. What would happen with a body on the equator, initially stationary relative to the planet's surface, if the period is shorter than that found in part (*a*)? How would it move?

B3 In the system shown, the two end particles (mass m) are connected to the central particle (mass M) with two springs (stiffness K). We consider 1-D motion only (x-axis). The coordinates for the displacements of each mass are x_1, x_2, and x_3. Find

a. the Lagrangian of the system.

b. the equation of motion.

c. the system's normal modes.



B4 Two balls of masses m and $3m$ collide head-on. The initial speed of the ball of mass m is v, and the second ball is initially at rest.

a. What are the final velocities of the balls, if the collision is purely elastic and one-dimensional?

b. Answer the same question if the collision is completely inelastic (the balls stick together). How much energy is dissipated in this case?

Physical Constants

speed of light .... $c = 2.998 \times 10^{8}$ m/s

Planck's constant .... $h = 6.626 \times 10^{-34}$ J·s

Planck's constant / 2π.... $\hbar = 1.055 \times 10^{-34}$ J·s

Boltzmann constant ...... $k_B = 1.381 \times 10^{-23}$ J/K

elementary charge .... $e = 1.602 \times 10^{-19}$ C

electric permittivity ...... $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m

magnetic permeability ... $\mu_0 = 1.257 \times 10^{-6}$ H/m

molar gas constant.......... $R = 8.314$ J / mol·K

Avogrado constant .... $N_A = 6.022 \times 10^{23}$ mol^{-1}

electrostatic constant ... $k = (4\pi\varepsilon_0)^{-1} = 8.988 \times 10^{9}$ m/F

electron mass .... $m_{el} = 9.109 \times 10^{-31}$ kg

electron rest energy...... 511.0 keV

Compton wavelength .. $\lambda_C = h / m_{el}c = 2.426$ pm

proton mass .... $m_p = 1.673 \times 10^{-27}\,\text{kg} = 1836 m_{el}$

1 bohr............................ $a_0 = \hbar^2 / ke^2 m_{el} = 0.5292$ Å

1 hartree (= 2 rydberg) ... $E_h = \hbar^2 / m_{el} a_0^{\,2} = 27.21$ eV

gravitational constant ... $G = 6.674 \times 10^{-11}$ m^3 / kg s^2

hc .... $hc = 1240$ eV·nm

Equations That May Be Helpful

THERMODYNAMICS

General efficiency η of a heat engine producing work $|W|$ while taking in heat $|Q_h|$ is $\eta = \dfrac{|W|}{|Q_h|}$.

For a Carnot cycle operating as a heat engine between reservoirs at T_h and at T_c the efficiency becomes $\eta_C = \dfrac{T_h - T_c}{T_h}$.

Clausius' theorem: $\sum_{i=1}^{N} \dfrac{Q_i}{T_i} \le 0$, which becomes $\sum_{i=1}^{N} \dfrac{Q_i}{T_i} = 0$ for a reversible cyclic process of N steps.

$$\frac{dp}{dT} = \frac{\lambda}{T\Delta V}$$

For adiabatic processes in an ideal gas with constant heat capacity, $pV^{\gamma} = \text{const.}$

$$dU = TdS - pdV \qquad dH = d(U + pV)$$
$$dF = d(U - TS) \qquad dG = d(U + pV - TS)$$

$$H = U + pV \qquad F = U - TS \qquad G = F + pV \qquad \Omega = F - \mu N$$

$$C_V = \left(\frac{\delta Q}{dT}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V \qquad C_p = \left(\frac{\delta Q}{dT}\right)_p = T\left(\frac{\partial S}{\partial T}\right)_p \qquad TdS = C_V dT + T\left(\frac{\partial S}{\partial V}\right)_T dV$$

Triple product: $\left(\frac{\partial X}{\partial Y}\right)_Z \cdot \left(\frac{\partial Y}{\partial Z}\right)_X \cdot \left(\frac{\partial Z}{\partial X}\right)_Y = -1$

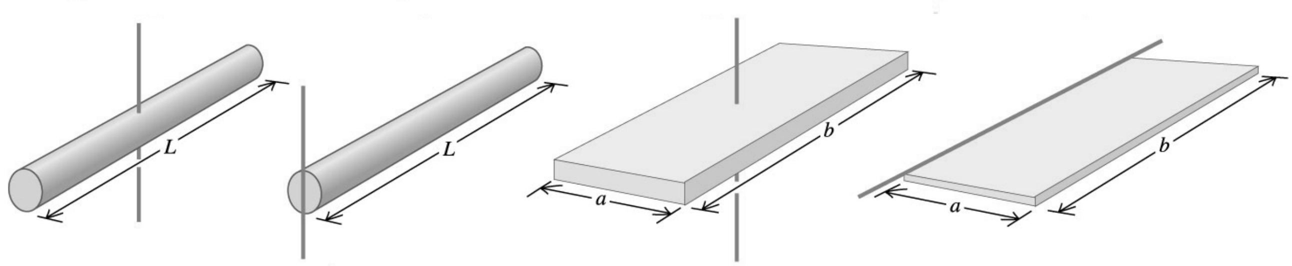
specific heat of water: 4186 J/(kg·K)
latent heat of ice melting: 334 J/g

MECHANICS

Gravitational acceleration at surface of Earth: $g = 9.81\ \text{m/s}^2$

Gauss's Law for gravity: $\oint_S \mathbf{g} \cdot d\mathbf{A} = -4\pi G m_{\text{encl.}}$

Moments of Inertia of Various Bodies

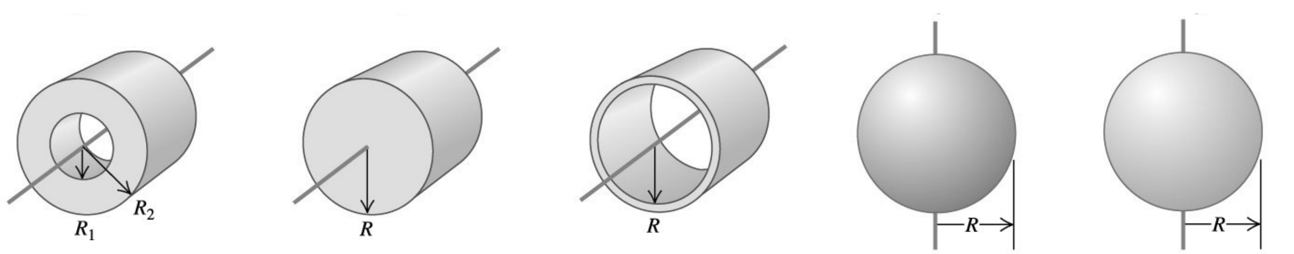


Slender rod, axis through center: $I = \frac{1}{12}ML^2$

Slender rod, axis through one end: $I = \frac{1}{3}ML^2$

Rectangular plate, axis through center: $I = \frac{1}{12}M(a^2 + b^2)$

Thin rectangular plate, axis along edge: $I = \frac{1}{3}Ma^2$



Hollow cylinder: $I = \frac{1}{2}M(R_1^2 + R_2^2)$

Solid cylinder: $I = \frac{1}{2}MR^2$

Thin-walled hollow cylinder: $I = MR^2$

Solid sphere: $I = \frac{2}{5}MR^2$

Thin-walled hollow sphere: $I = \frac{2}{3}MR^2$

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient: $\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

Curl: $\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$

Laplacian: $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient: $\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$

Curl:

$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian: $\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient: $\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$

Curl: $\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$

Laplacian: $\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem: $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

INTEGRALS

$f(x)$	$\int_0^\infty f(x)\,dx$
e^{-ax^2}	$\frac{\sqrt{\pi}}{2\sqrt{a}}$
xe^{-ax^2}	$\frac{1}{2a}$
$x^2e^{-ax^2}$	$\frac{\sqrt{\pi}}{4a^{3/2}}$
$x^3e^{-ax^2}$	$\frac{1}{2a^2}$
$x^4e^{-ax^2}$	$\frac{3\sqrt{\pi}}{8a^{5/2}}$
$x^5e^{-ax^2}$	$\frac{1}{a^3}$
$x^6e^{-ax^2}$	$\frac{15\sqrt{\pi}}{16a^{7/2}}$

$$\int_0^\infty \frac{1}{1+ay^2}dy = \pi / 2a^{1/2}$$

$$\int_0^\infty y^n e^{-y/a}dy = n!a^{n+1}$$

$$\int x^{-2}\ln(x)\,dx = -\frac{\ln(x)}{x} - \frac{1}{x}$$

$$\int x^{-1}\ln(x)\,dx = \tfrac{1}{2}\ln^2(x)$$

$$\int \ln(x)\,dx = x\ln(x) - x$$

$$\int x\ln(x)\,dx = \tfrac{1}{2}x^2\ln(x) - \tfrac{1}{4}x^2$$

$$\int x^2\ln(x)\,dx = \tfrac{1}{3}x^3\ln(x) - \tfrac{1}{9}x^3$$

$$\int \ln(1+x)\,dx = (x+1)\big(\ln(x+1)-1\big) + C$$

$$\int \frac{r^3dr}{(x^2+r^2)^{3/2}} = (r^2+x^2)^{1/2} + \frac{x^2}{(r^2+x^2)^{1/2}}$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right); \quad \left|\tan^{-1}\left(\frac{x}{a}\right)\right| < \frac{\pi}{2}$$

$$\int \frac{xdx}{a^2+x^2} = \frac{1}{2}\ln\left(a^2+x^2\right)$$

$$\int \frac{dx}{x(a^2+x^2)} = \frac{1}{2a^2}\ln\left(\frac{x^2}{a^2+x^2}\right)$$

$$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right)$$

$$= -\frac{1}{ab}\coth^{-1}\left(\frac{ax}{b}\right)$$

$$= -\frac{1}{ab}\tanh^{-1}\left(\frac{ax}{b}\right); \quad a^2x^2 < b^2$$

POWER SERIES

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \quad \left(|x| \le 1,\ x \ne -1\right)$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \quad \left(|x| < 1\right)$$