UNL - Department of Physics and Astronomy

Preliminary Examination - Day 1
Thursday, August 10, 2017

This test covers the topics of *Quantum Mechanics* (Topic 1) and *Electrodynamics* (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

Note: If you do more than two problems in a group, only the first two (in the order they appear in this handout) will be graded. For instance, if you do problems A1, A3, and A4, only A1 and A3 will be graded.

WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY

Quantum Mechanics Group A - Answer only two Group A questions

A1 The operator $\hat{R}$ is defined by $\hat{R}\psi(x) = \mathrm{Re}[\psi(x)]$. Is $\hat{R}$ a linear operator? Explain.

A2 Find the energy levels of a spin $s = \frac{3}{2}$ particle whose Hamiltonian is given by

$\hat{H} = \frac{\alpha}{\hbar^2}\left(\hat{S}_x^{\ 2} + \hat{S}_y^{\ 2} - 2\hat{S}_z^{\ 2}\right) - \frac{\beta}{\hbar}\hat{S}_z$, where α and β are constants.

A3 A 40 keV photon collides with an electron at rest. The photon is scattered at 90°.

(a) What is the incoming photon's wavelength?

(b) What is the kinetic energy of the recoil electron?

(c) What is the wavelength of the recoil electron?

A4 A particle of mass m and kinetic energy E is incident on a potential barrier of the form

$$V(x) = 0, \quad x < 0$$
$$V(x) = V_0, \quad x > 0$$

with $V_0 < E$.

(a) Find the transmission and reflection coefficients T and R.

(b) Show that the obtained expressions satisfy the conservation of probability law.

<u>Quantum Mechanics Group B</u> <u>- Answer only two Group B questions</u>

B1 Consider a system which is initially in the normalized state

$$\psi(\theta,\phi) = \frac{1}{\sqrt{5}} Y_{1,-1}(\theta,\phi) + a Y_{1,0}(\theta,\phi) + \frac{1}{\sqrt{5}} Y_{1,1}(\theta,\phi)$$

in which a is a positive real constant.

a. Find *a*.

b. If L_z were measured, what values could one obtain, and with what probabilities?

We now measure L_z and find the value $-\hbar$.

c. Calculate $\langle L_x \rangle$ and $\langle L_y \rangle$.

d. Calculate the uncertainties ΔL_x and ΔL_y, and their product $\Delta L_x \Delta L_y$. You may use the equality $\langle L_x{}^2 \rangle = \langle L_y{}^2 \rangle$ without proving it first.

B2 The Hamiltonian for a one-dimensional harmonic oscillator is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \tfrac{1}{2} m\omega^2 \hat{x}^2 .$$

We write its energy eigenkets as $|n\rangle$ ($n = 0,1,2,\ldots$) for energy $E_n = (n + \tfrac{1}{2})\hbar\omega$.

a. Suppose the system is in the normalized state $|\varphi\rangle$ given by $|\varphi\rangle = c_0 |0\rangle + c_1 |1\rangle$, and that the expectation value of the energy is known to be $\hbar\omega$. What are $|c_0|$ and $|c_1|$?

b. Now choose c_0 to be real and positive, but let c_1 have any phase: $c_1 = |c_1| e^{i\theta}$. Suppose further that not only is the expectation value of the energy known to be $\hbar\omega$, but the expectation value of x is also known: $\langle \varphi | x | \varphi \rangle = \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}$. Calculate the phase angle θ.

c. Now suppose the system is in the state $|\varphi\rangle$ at time $t = 0$, i.e., $|\psi(t=0)\rangle = |\varphi\rangle$. Calculate $|\psi(t)\rangle$ at some later time t. Use the values of c_0 and c_1 you found in parts *a.* and *b.*

d. Also calculate the expectation value of x as a function of time. With what angular frequency does it oscillate? Again, use the values of c_0 and c_1 you found in parts *a.* and *b.*

B3 The eigenstates of a particle in a box are given by

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{\pi n x}{L}, n = 1, 2, 3, ...$$

if the walls are at the positions $x = 0$ and $x = L$.

(a) Rewrite the same eigenfunctions for the case when the walls are at the positions $x = -L/2$ and $x = L/2$.

(b) Are these functions parity eigenstates? If yes, what are the parity eigenvalues?

(c) Are these functions momentum eigenstates? If yes, what are the momentum eigenvalues?

(d) Calculate the expectation values of p and p^2 for the eigenstate n.

(e) Find the uncertainty Δp and interpret your result by representing each eigenstate as a superposition of two exponentials.

(f) Calculate the expectation value of xp for the ground state. Could you conclude from your answer if this operator is Hermitian?

B4 A particle at rest with spin 1/2 and gyromagnetic ratio γ is placed in a magnetic field B directed along x axis.

(a) Write down the Schrödinger equation describing the evolution of the particle's wavefunction.

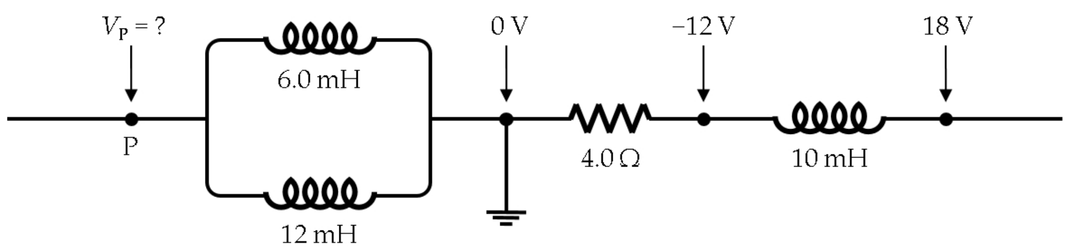
(b) Solve it assuming that at $t = 0$ the particle is in eigenstate of S_z with the eigenvalue $\hbar/2$.

(c) Calculate expectation values of S_z and S_y as functions of t.

Electrodynamics Group A - Answer only two Group A questions

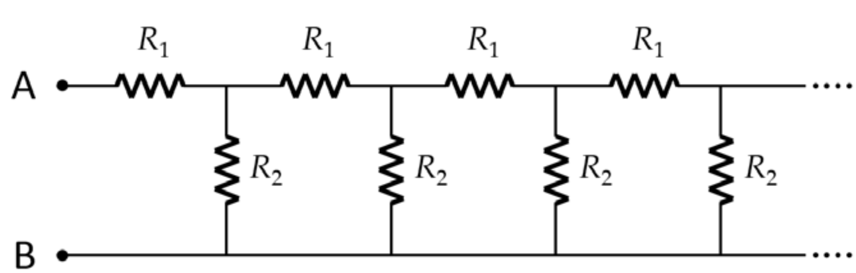
A1 An isolated sphere of perfectly conducting material is surrounded by air. Though normally a good insulator, air breaks down (it becomes conductive) for electric fields beyond 3.0 kV/mm (the so-called *dielectric strength* of air). The sphere's radius is 5.0 cm. What is the maximum amount of electrostatic energy the sphere can store before breakdown occurs? Assume the electrostatic potential is zero at infinite distance from the sphere.

A2 The diagram shows part of an electronic circuit. Calculate the potential at point P.



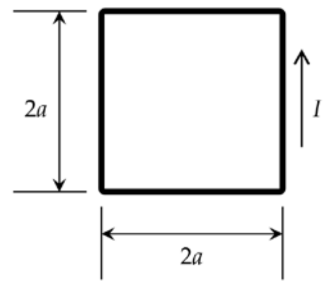
A3 If the electric field generated by a static point charge were proportional to $1/r^3$, where r is the distance from the charge, would Gauss' law still be correct? Justify your answer.

A4 The diagram shows an infinitely long chain of resistors. What is the resistance between points A and B? *Hint: If the chain's length is increased by one unit, how does this resistance change?*



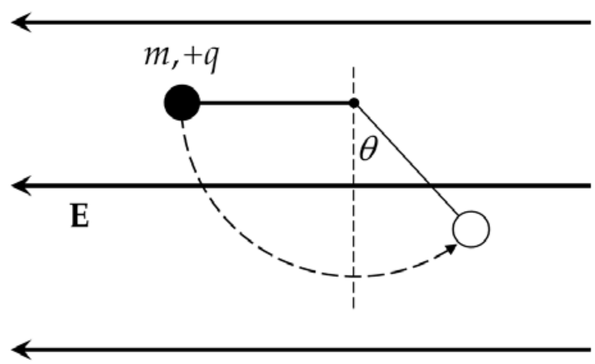
Electrodynamics Group B - Answer only two Group B questions

B1 A flat square loop of wire of length $2a$ on each side carries a stationary current I. Calculate the magnitude of the magnetic field at the center of the square.



B2 All space is filled with a material with uniform, fixed magnetization $\mathbf{M}$, except for the region $0 < z < a$, in which there is vacuum. The magnetization is $\mathbf{M} = M\hat{\mathbf{u}}$, where $\hat{\mathbf{u}}$ is a unit vector in the yz plane that makes an angle θ with the z-axis: $\hat{\mathbf{u}} = (\sin\theta)\hat{\mathbf{y}} + (\cos\theta)\hat{\mathbf{z}}$. Calculate the magnetic field $\mathbf{B}$ and the auxiliary field $\mathbf{H}$ everywhere.

B3 A small ball with mass m and electric charge $+q$ is hung in a horizontal, uniform electric field $\mathbf{E}$ by a string of negligible mass. The ball is raised to the position shown in the figure and then dropped from rest. At what angle θ will it come to rest again?



B4 An infinitely large uniform slab of linear, isotropic dielectric material of permittivity ε is parallel to the xy plane. It is exposed to an external electric field $\mathbf{E}_0$ perpendicular to the slab (so, in the z direction). Find the polarization $\mathbf{P}$ inside the slab.

Physical Constants

speed of light .... $c = 2.998 \times 10^{8}$ m/s	electrostatic constant ... $k = (4\pi\varepsilon_0)^{-1} = 8.988 \times 10^{9}$ m/F
Planck's constant .... $h = 6.626 \times 10^{-34}$ J·s	electron mass .... $m_{\mathrm{el}} = 9.109 \times 10^{-31}$ kg
Planck's constant / 2π.... $\hbar = 1.055 \times 10^{-34}$ J·s	electron rest energy...... 511.0 keV
Boltzmann constant ...... $k_{\mathrm{B}} = 1.381 \times 10^{-23}$ J/K	Compton wavelength .. $\lambda_{\mathrm{C}} = h/m_{\mathrm{el}}c = 2.426$ pm
elementary charge .... $e = 1.602 \times 10^{-19}$ C	proton mass .... $m_{\mathrm{p}} = 1.673 \times 10^{-27}\,\mathrm{kg} = 1836 m_{\mathrm{el}}$
electric permittivity ...... $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m	1 bohr .... $a_0 = \hbar^2/ke^2 m_{\mathrm{el}} = 0.5292$ Å
magnetic permeability ... $\mu_0 = 1.257 \times 10^{-6}$ H/m	1 hartree (= 2 rydberg) ... $E_{\mathrm{h}} = \hbar^2/m_{\mathrm{el}}a_0^{\,2} = 27.21$ eV
molar gas constant.......... $R = 8.314$ J / mol·K	gravitational constant ... $G = 6.674 \times 10^{-11}$ m^3/ kg s^2
Avogadro constant .... $N_{\mathrm{A}} = 6.022 \times 10^{23}$ mol^{-1}	hc .... $hc = 1240$ eV·nm

Equations That May Be Helpful

TRIGONOMETRY

$$\sin\alpha\sin\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$
$$\cos\alpha\cos\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$
$$\sin\alpha\cos\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$
$$\cos\alpha\sin\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$$

QUANTUM MECHANICS

Ground-state wavefunction of the hydrogen atom: $\psi(\mathbf{r}) = \dfrac{e^{-r/a_0}}{\pi^{1/2} a_0^{3/2}}$, where $a_0 = \dfrac{4\pi\varepsilon_0\hbar^2}{me^2}$ is the Bohr radius, using $m \approx m_{\mathrm{el}}$, in which m_{el} is the electron mass.

$$\psi_{nlm}(\mathbf{r}) = R_{nl}(r)Y_{lm}(\hat{\mathbf{r}})$$
$$R_{10}(r) = \frac{2}{a_0^{3/2}} e^{-r/a_0}$$
$$R_{21}(r) = \frac{1}{3^{1/2}(2a_0)^{3/2}} \frac{r}{a_0} e^{-r/2a_0}$$
$$E_n = -\frac{1}{n^2}\frac{mk^2e^4}{2\hbar^2}$$

Particle in one-dimensional, infinitely-deep box with walls at $x=0$ and $x=a$:

Stationary states $\psi_n = (2/a)^{1/2}\sin(n\pi x/a)$, energy levels $E_n = n^2 \dfrac{\pi^2\hbar^2}{2ma^2}$

Angular momentum: $[L_x, L_y] = i\hbar L_z$ *et cycl.*

Ladder operators:
$$L_+ |\ell,m\rangle = \hbar\sqrt{(\ell+m+1)(\ell-m)}\,|\ell,m+1\rangle$$
$$L_- |\ell,m\rangle = \hbar\sqrt{(\ell+m)(\ell-m+1)}\,|\ell,m-1\rangle$$

Creation, annihilation operators:
$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right) \qquad \hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right)$$
$$\hat{a}^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle \qquad \hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$$

Table Spherical harmonics and their expressions in Cartesian coordinates.

$Y_{lm}(\theta,\varphi)$	$Y_{lm}(x,y,z)$
$Y_{00}(\theta,\varphi) = \frac{1}{\sqrt{4\pi}}$	$Y_{00}(x,y,z) = \frac{1}{\sqrt{4\pi}}$
$Y_{10}(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta$	$Y_{10}(x,y,z) = \sqrt{\frac{3}{4\pi}}\frac{z}{r}$
$Y_{1,\pm1}(\theta,\varphi) = \mp\sqrt{\frac{3}{8\pi}}e^{\pm i\varphi}\sin\theta$	$Y_{1,\pm1}(x,y,z) = \mp\sqrt{\frac{3}{8\pi}}\frac{x\pm iy}{r}$
$Y_{20}(\theta,\varphi) = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$	$Y_{20}(x,y,z) = \sqrt{\frac{5}{16\pi}}\frac{3z^2-r^2}{r^2}$
$Y_{2,\pm1}(\theta,\varphi) = \mp\sqrt{\frac{15}{8\pi}}e^{\pm i\varphi}\sin\theta\cos\theta$	$Y_{2,\pm1}(x,y,z) = \mp\sqrt{\frac{15}{8\pi}}\frac{(x\pm iy)z}{r^2}$
$Y_{2,\pm2}(\theta,\varphi) = \sqrt{\frac{15}{32\pi}}e^{\pm 2i\varphi}\sin^2\theta$	$Y_{2,\pm2}(x,y,z) = \mp\sqrt{\frac{15}{32\pi}}\frac{x^2-y^2\pm 2ixy}{r^2}$

$H_{\text{mag}} = -\gamma\,\mathbf{S}\cdot\mathbf{B}$

Pauli matrices: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Compton scattering: $\lambda' - \lambda = \lambda_C(1-\cos\theta)$

ELECTROSTATICS

$\oiint_S \mathbf{E}\cdot\hat{\mathbf{n}}\,da = \frac{q_{\text{encl}}}{\varepsilon_0}$ $\qquad \mathbf{E} = -\nabla V$ $\qquad \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{E}\cdot d\boldsymbol{\ell} = V(\mathbf{r}_1) - V(\mathbf{r}_2)$ $\qquad V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$

Work done $W = -\int_{\mathbf{a}}^{\mathbf{b}} q\mathbf{E}\cdot d\boldsymbol{\ell} = q\left[V(\mathbf{b}) - V(\mathbf{a})\right]$ Energy stored in elec. field: $W = \frac{1}{2}\varepsilon_0\int_V E^2 d\tau = Q^2/2C$

Relative permittivity: $\varepsilon_{\text{r}} = 1 + \chi_{\text{e}}$

Bound charges

$\rho_{\text{b}} = -\nabla\cdot\mathbf{P}$
$\sigma_{\text{b}} = \mathbf{P}\cdot\hat{\mathbf{n}}$

Capacitance in vacuum

Parallel-plate: $C = \varepsilon_0 \frac{A}{d}$

Spherical: $C = 4\pi\varepsilon_0 \frac{ab}{b-a}$

Cylindrical: $C = 2\pi\varepsilon_0 \frac{L}{\ln(b/a)}$ (for a length L)

MAGNETOSTATICS

Relative permeability: $\mu_{\text{r}} = 1 + \chi_{\text{m}}$

Lorentz Force: $\mathbf{F} = q\mathbf{E} + q(\mathbf{v}\times\mathbf{B})$ Current densities: $I = \int \mathbf{J}\cdot d\mathbf{A}$, $I = \int \mathbf{K}\cdot d\boldsymbol{\ell}$

Biot-Savart Law: $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{Id\boldsymbol{\ell}\times\hat{\mathbf{R}}}{R^2}$ ($\mathbf{R}$ is vector from source point to field point $\mathbf{r}$)

Infinitely long solenoid: B-field inside is $B = \mu_0 nI$ (n is number of turns per unit length)

Ampere's law: $\oint \mathbf{B}\cdot d\boldsymbol{\ell} = \mu_0 I_{\text{encl}}$

Magnetic dipole moment of a current distribution is given by $\mathbf{m} = I\int d\mathbf{a}$.

Force on magnetic dipole: $\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$

Torque on magnetic dipole: $\boldsymbol{\tau} = \mathbf{m}\times\mathbf{B}$

B-field of magnetic dipole: $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\right]$

Bound currents

$J_b = \nabla \times \mathbf{M}$
$K_b = \mathbf{M} \times \hat{\mathbf{n}}$

Maxwell's Equations in vacuum

1.	$\nabla \cdot \mathbf{E} = \dfrac{\rho}{\varepsilon_0}$	Gauss' Law
2.	$\nabla \cdot \mathbf{B} = 0$	no magnetic charge
3.	$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	Faraday's Law
4.	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \varepsilon_0 \mu_0 \dfrac{\partial \mathbf{E}}{\partial t}$	Ampere's Law with Maxwell's correction

Maxwell's Equations in linear, isotropic, and homogeneous (LIH) media

1.	$\nabla \cdot \mathbf{D} = \rho_f$	Gauss' Law
2.	$\nabla \cdot \mathbf{B} = 0$	no magnetic charge
3.	$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	Faraday's Law
4.	$\nabla \times \mathbf{H} = \mathbf{J}_f + \dfrac{\partial \mathbf{D}}{\partial t}$	Ampere's Law with Maxwell's correction

Induction

Alternative way of writing Faraday's Law: $\oint \mathbf{E} \cdot d\boldsymbol{\ell} = -\dfrac{d\Phi_B}{dt}$

Mutual and self inductance: $\Phi_2 = M_{21} I_1$, and $M_{21} = M_{12}$; $\Phi = LI$

Energy stored in magnetic field: $W = \frac{1}{2} \mu_0^{-1} \int_V B^2 d\tau = \frac{1}{2} LI^2 = \frac{1}{2} \oint \mathbf{A} \cdot \mathbf{I} d\ell$

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$; $d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla\cdot\mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla\times\mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$; $d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

Curl : $$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$; $d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla\cdot\mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla\times\mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

CARTESIAN AND SPHERICAL UNIT VECTORS

$$\hat{\mathbf{x}} = (\sin\theta\cos\phi)\hat{\mathbf{r}} + (\cos\theta\cos\phi)\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{y}} = (\sin\theta\sin\phi)\hat{\mathbf{r}} + (\cos\theta\sin\phi)\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

INTEGRALS

$f(x)$	$\int_0^\infty f(x)\,dx$
e^{-ax^2}	$\dfrac{\sqrt{\pi}}{2\sqrt{a}}$
xe^{-ax^2}	$\dfrac{1}{2a}$
$x^2e^{-ax^2}$	$\dfrac{\sqrt{\pi}}{4a^{3/2}}$
$x^3e^{-ax^2}$	$\dfrac{1}{2a^2}$
$x^4e^{-ax^2}$	$\dfrac{3\sqrt{\pi}}{8a^{5/2}}$
$x^5e^{-ax^2}$	$\dfrac{1}{a^3}$
$x^6e^{-ax^2}$	$\dfrac{15\sqrt{\pi}}{16a^{7/2}}$

$$\int_0^\infty \frac{1}{1+bx^2}dx = \pi/2b^{1/2}$$
$$\int_0^\infty x^n e^{-bx}dx = \frac{n!}{b^{n+1}}$$
$$\int (x^2+b^2)^{-1/2}dx = \ln\left(x+\sqrt{x^2+b^2}\right)$$
$$\int (x^2+b^2)^{-1}dx = \frac{1}{b}\arctan(x/b)$$
$$\int (x^2+b^2)^{-3/2}dx = \frac{x}{b^2\sqrt{x^2+b^2}}$$
$$\int (x^2+b^2)^{-2}dx = \frac{\dfrac{bx}{x^2+b^2}+\arctan(x/b)}{2b^3}$$
$$\int \frac{x\,dx}{x^2+b^2} = \frac{1}{2}\ln\left(x^2+b^2\right)$$
$$\int \frac{dx}{x(x^2+b^2)} = \frac{1}{2b^2}\ln\left(\frac{x^2}{x^2+b^2}\right)$$
$$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right)$$
$$= -\frac{1}{ab}\text{artanh}\left(\frac{ax}{b}\right)$$