UNL - Department of Physics and Astronomy

Preliminary Examination - Day 2
Friday, August 11, 2017

This test covers the topics of *Thermodynamics and Statistical Mechanics* (Topic 1) and *Mechanics* (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

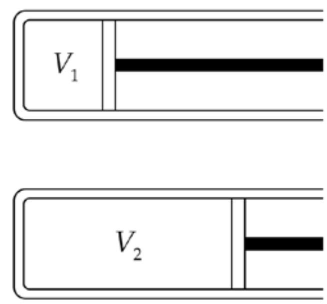
Note: If you do more than two problems in a group, only the first two (in the order they appear in this handout) will be graded. For instance, if you do problems A1, A3, and A4, only A1 and A3 will be graded.

WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY

Thermodynamics and Statistical Mechanics Group A - Answer only two Group A questions

A1 You do 25 kJ of work on a system consisting of 3.0 kg of water by stirring it with a paddle wheel. During this time, 63 kJ of heat is released by the system due to poor thermal insulation. What is the change in the internal energy of the system?

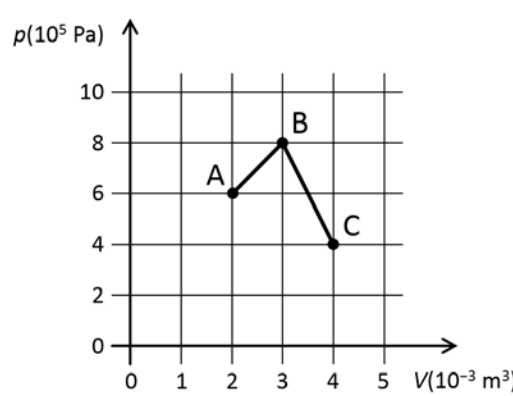
A2 N atoms of a perfect gas are contained in a cylinder with insulating walls, closed at one end by a piston. The initial volume is V_1 and the initial temperature T_1. Find the change in entropy that would occur if the volume were suddenly increased to V_2 by withdrawing the piston.



A3 Compression in a diesel engine occurs quickly enough so that very little heating of the environment occurs, and thus the process may be considered adiabatic. If a temperature of 500°C is required for ignition, what is the compression ratio? Assume that the air can be treated as an ideal gas with $\gamma = 1.4$, and the temperature is 20°C before compression.

A4 The internal energy of some ideal gas is given by $E = \frac{5}{2}nRT$. One mole of this gas is taken quasistatically from state A to state B, and then from state B to state C, along the paths shown in the figure.

a. What is the molar heat capacity at constant volume?
b. What is the work done by the gas in the process $A \to B \to C$?
c. What is the heat absorbed by the gas in the process?
d. What is the change of entropy in the process?



Thermodynamics and Statistical Mechanics Group B - Answer only two Group B questions

B1 This probability question is known to be asked at Wall Street interviews. The original formulation is kept. "Let's play a game of Russian roulette. You are tied to your chair. Here's a gun, a revolver. Here's the barrel of the gun, six chambers, all empty. Now watch me as I put two bullets into the barrel, into two adjacent chambers. I close the barrel and spin it. I put the gun to your head and pull the trigger. Click. Lucky you! Now I'm going to pull the trigger one more time. Which would you prefer: that I spin the barrel first or that I just pull the trigger?" *Note: If the barrel is not spun, it will simply move to the next chamber.*

B2 Given are 1.0 kg of water at 100°C and a very large block of ice at 0°C. A reversible heat engine absorbs heat from the water and expels heat to the ice until work can no longer be extracted from the system. At the completion of the process, how much work has been done by the engine?

B3 An experimentalist determines that the heat capacity of a substance obeys the empirical relation $C_V(T,V) = \alpha T^2 V^3$, where α is a constant. The experimentalist also finds the entropy and energy to be zero at absolute zero for all volumes. Find the expression for the Helmholtz free energy $F(T,V)$ for a system with a fixed number of particles.

B4 The free expansion of a gas is a process where the total energy E is constant. Find the following quantities:

a. $(\partial T/\partial V)_E$ in terms of p, T, $(\partial p/\partial T)_V$, and C_V;

b. $(\partial S/\partial V)_E$ in terms of p and T.

c. Using parts *a.* and *b.*, calculate the temperature change in a free expansion from V_1 to V_2 for a van der Waals gas.

Mechanics Group A - Answer only two Group A questions

A1 A particle of mass m is performing one-dimensional motion subject to the force

$$F(x) = -Ax + Bx^3$$

where A and B are positive constants.

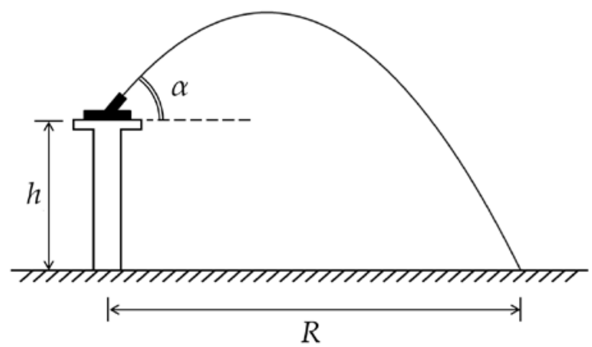
At some instant the particle's position is $x = 0$ and the velocity $v = v_0$.

(a) Find the potential energy as a function of x and sketch it. Indicate the positions of all maxima and minima.

(b) Find the velocity as a function of x.

(c) Find the condition on the initial velocity v_0 for which the motion is periodic ($v_0 <?$).

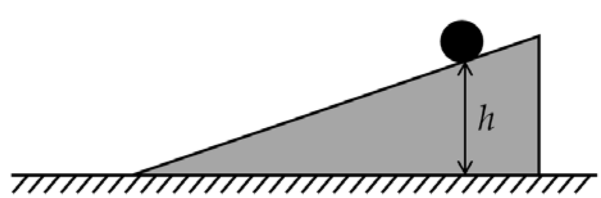
A2 A cannon with elevation angle α is mounted on a vertical tower of height h that overlooks a level plain below. What should be the initial speed v_0 of the shell in order to achieve the range R? Neglect air resistance.



A3 A rocket of mass 5000 kg is launched vertically. The ejected gas has speed 1000 m/s. What is the necessary rate of ejection (mass per time) …

a. … to support the weight of the rocket?
b. … to give it an upward acceleration of $2g$?

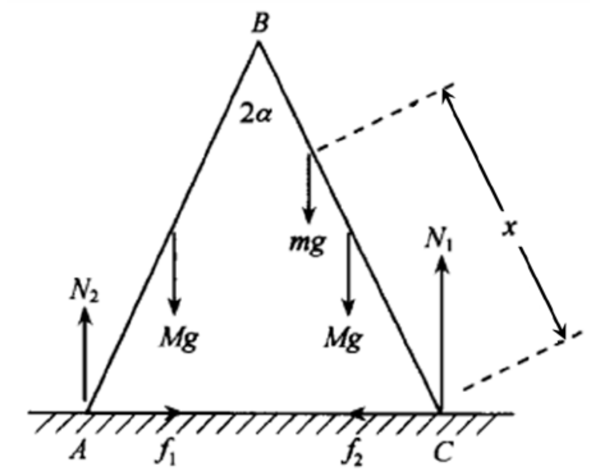
A4 A uniform cylinder of mass m and radius R rolls along an inclined plane. Find the speed of its center at the bottom if the cylinder is initially released from rest at height h. There is no slipping.



Mechanics Group B - Answer only two Group B questions

B1 A circular ring of mass M and radius R lies at rest on a frictionless, horizontal surface. An insect of mass m sits on it. The insects starts crawling round the ring, with constant speed v relative to the ring. Find the angular velocity of the ring.

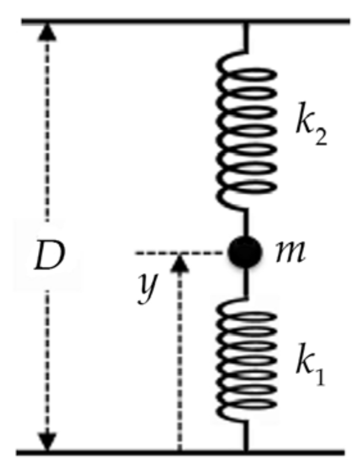
B2 Two identical, uniform ladders with length $2a$ and mass M are connected via a frictionless hinge and put on the ground, as shown in the figure. The coefficient of static friction between the ladders and the ground is μ, and the angle between the two ladders is 2α. What coefficient of static friction (μ) is required so that a person with mass m can climb to the top from either side safely?



B3 Suppose that the Earth suddenly stopped orbiting the Sun. Find the time it would take for the Earth to fall into the Sun. You must give a numerical answer.

B4 As shown in the figure, a block with mass m is hung between two springs with spring constants k_1 and k_2 and equilibrium lengths d_1 and d_2 (with $d_1 + d_2 < D$). The mass of each spring is negligible, and the block can only move in the vertical direction. The acceleration due to gravity is g.

a. Write down the Lagrangian.
b. Find the equation of motion.
c. Find the value of y for which there is equilibrium.



Physical Constants

speed of light.................. $c = 2.998 \times 10^{8}$ m/s

Planck's constant .... $h = 6.626 \times 10^{-34}$ J·s

Planck's constant / 2π.... $\hbar = 1.055 \times 10^{-34}$ J·s

Boltzmann constant ...... $k_B = 1.381 \times 10^{-23}$ J/K

elementary charge .... $e = 1.602 \times 10^{-19}$ C

electric permittivity ...... $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m

magnetic permeability ... $\mu_0 = 1.257 \times 10^{-6}$ H/m

molar gas constant.......... $R = 8.314$ J / mol·K

Avogadro constant .... $N_A = 6.022 \times 10^{23}$ mol^{-1}

electrostatic constant ... $k = (4\pi\varepsilon_0)^{-1} = 8.988 \times 10^{9}$ m/F

electron mass .... $m_{el} = 9.109 \times 10^{-31}$ kg

electron rest energy...... 511.0 keV

Compton wavelength .. $\lambda_C = h / m_{el}c = 2.426$ pm

proton mass .... $m_p = 1.673 \times 10^{-27}\,\text{kg} = 1836 m_{el}$

1 bohr............................ $a_0 = \hbar^2 / ke^2 m_{el} = 0.5292$ Å

1 hartree (= 2 rydberg) ... $E_h = \hbar^2 / m_{el} a_0^{\,2} = 27.21$ eV

gravitational constant ... $G = 6.674 \times 10^{-11}$ m^3/ kg s^2

hc .... $hc = 1240$ eV·nm

Equations That May Be Helpful

POWER SERIES

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \quad \left(|x| \le 1,\ x \ne -1\right)$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \quad \left(|x| < 1\right)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots$$

THERMODYNAMICS

General efficiency η of a heat engine producing work $|W|$ while taking in heat $|Q_h|$ is $\eta = \frac{|W|}{|Q_h|}$.

For a Carnot cycle operating as a heat engine between reservoirs at T_h and at T_c the efficiency becomes $\eta_C = \frac{T_h - T_c}{T_h}$.

Clausius' theorem: $\sum_{i=1}^{N} \frac{Q_i}{T_i} \leq 0$, which becomes $\sum_{i=1}^{N} \frac{Q_i}{T_i} = 0$ for a reversible cyclic process of N steps.

$$\frac{dp}{dT} = \frac{\lambda}{T\Delta V}$$

For adiabatic processes in an ideal gas with constant heat capacity, $pV^{\gamma} = \text{const.}$

$$dU = TdS - pdV \qquad dH = d(U + pV)$$
$$dF = d(U - TS) \qquad dG = d(U + pV - TS)$$

$$H = U + pV \qquad F = U - TS \qquad G = F + pV \qquad \Omega = F - \mu N$$

$$C_V = \left(\frac{\delta Q}{dT}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V \qquad C_p = \left(\frac{\delta Q}{dT}\right)_p = T\left(\frac{\partial S}{\partial T}\right)_p \qquad TdS = C_V dT + T\left(\frac{\partial S}{\partial V}\right)_T dV$$

$$\kappa = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T \qquad \alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p$$

Triple product: $\left(\frac{\partial X}{\partial Y}\right)_Z \cdot \left(\frac{\partial Y}{\partial Z}\right)_X \cdot \left(\frac{\partial Z}{\partial X}\right)_Y = -1$

specific heat of water: 4186 J/(kg·K)
latent heat of ice melting: 334 J/g

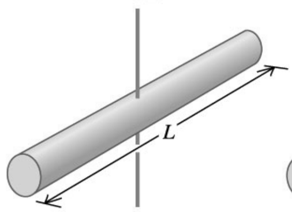
Van der Waals gas: $p = \frac{nRT}{V - bn} - \frac{n^2 a}{V^2}$, where n is the number of moles of gas

MECHANICS

Gravitational acceleration at surface of Earth: $g = 9.81\ \text{m/s}^2$

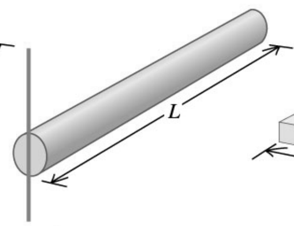
Gauss's Law for gravity: $\oint_S \mathbf{g} \cdot d\mathbf{A} = -4\pi G m_{\text{encl.}}$

Moments of Inertia of Various Bodies



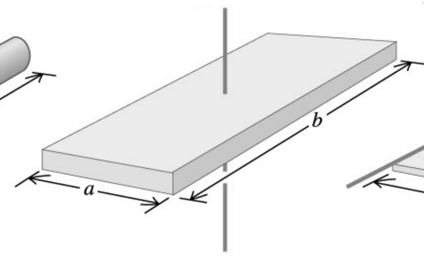
Slender rod,
axis through center

$I = \frac{1}{12}ML^2$



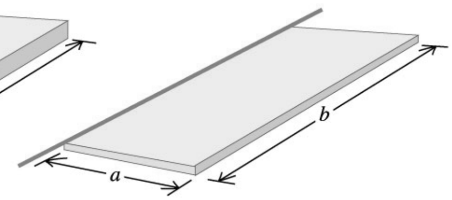
Slender rod,
axis through one end

$I = \frac{1}{3}ML^2$



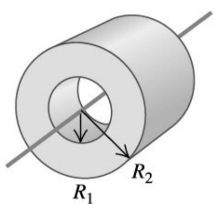
Rectangular plate,
axis through center

$I = \frac{1}{12}M\left(a^2 + b^2\right)$



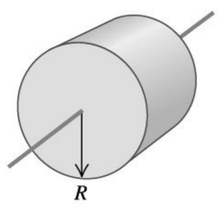
Thin rectangular plate,
axis along edge

$I = \frac{1}{3}Ma^2$



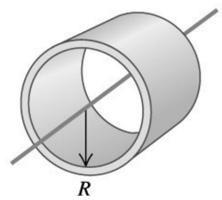
Hollow cylinder

$I = \frac{1}{2}M\left(R_1^2 + R_2^2\right)$



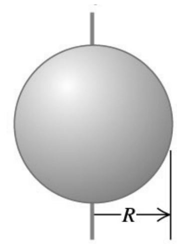
Solid cylinder

$I = \frac{1}{2}MR^2$



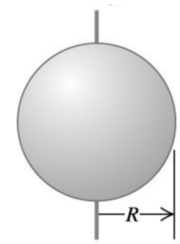
Thin-walled hollow
cylinder

$I = MR^2$



Solid sphere

$I = \frac{2}{5}MR^2$



Thin-walled hollow
sphere

$I = \frac{2}{3}MR^2$

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence : $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

Curl : $\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$

Laplacian : $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$

Divergence : $\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$

Curl :
$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence : $\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$

Curl : $\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$

Laplacian : $\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

INTEGRALS

$f(x)$	$\int_0^\infty f(x)\,dx$
e^{-ax^2} ……………	$\dfrac{\sqrt{\pi}}{2\sqrt{a}}$
xe^{-ax^2} ……………	$\dfrac{1}{2a}$
$x^2e^{-ax^2}$ ……………	$\dfrac{\sqrt{\pi}}{4a^{3/2}}$
$x^3e^{-ax^2}$ ……………	$\dfrac{1}{2a^2}$
$x^4e^{-ax^2}$ ……………	$\dfrac{3\sqrt{\pi}}{8a^{5/2}}$
$x^5e^{-ax^2}$ ……………	$\dfrac{1}{a^3}$
$x^6e^{-ax^2}$ ……………	$\dfrac{15\sqrt{\pi}}{16a^{7/2}}$

$$\int_0^\infty \frac{1}{1+ay^2}\,dy = \pi / 2a^{1/2}$$

$$\int_0^\infty y^n e^{-y/a}\,dy = n!\,a^{n+1}$$

$$\int_0^a \frac{dx}{\sqrt{(1/x)-(1/a)}} = \tfrac{1}{2}\pi a^{3/2}$$

$$\begin{aligned}
\int x^{-2}\ln(x)\,dx &= -\frac{\ln(x)}{x} - \frac{1}{x} \\
\int x^{-1}\ln(x)\,dx &= \tfrac{1}{2}\ln^2(x) \\
\int \ln(x)\,dx &= x\ln(x) - x \\
\int x\ln(x)\,dx &= \tfrac{1}{2}x^2\ln(x) - \tfrac{1}{4}x^2 \\
\int x^2\ln(x)\,dx &= \tfrac{1}{3}x^3\ln(x) - \tfrac{1}{9}x^3
\end{aligned}$$

$$\int \ln(1+x)\,dx = (x+1)\big(\ln(x+1)-1\big)+C$$

$$\begin{aligned}
\int \frac{r^3\,dr}{(x^2+r^2)^{3/2}} &= (r^2+x^2)^{1/2} + \frac{x^2}{(r^2+x^2)^{1/2}} \\
\int \frac{dx}{a^2+x^2} &= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) \qquad \left|\tan^{-1}\left(\frac{x}{a}\right)\right| < \frac{\pi}{2} \\
\int \frac{x\,dx}{a^2+x^2} &= \frac{1}{2}\ln\left(a^2+x^2\right) \\
\int \frac{dx}{x(a^2+x^2)} &= \frac{1}{2a^2}\ln\left(\frac{x^2}{a^2+x^2}\right) \\
\int \frac{dx}{a^2x^2-b^2} &= \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right) \\
&= -\frac{1}{ab}\coth^{-1}\left(\frac{ax}{b}\right) \\
&= -\frac{1}{ab}\tanh^{-1}\left(\frac{ax}{b}\right), \quad a^2x^2 < b^2
\end{aligned}$$