UNL Department of Physics and Astronomy

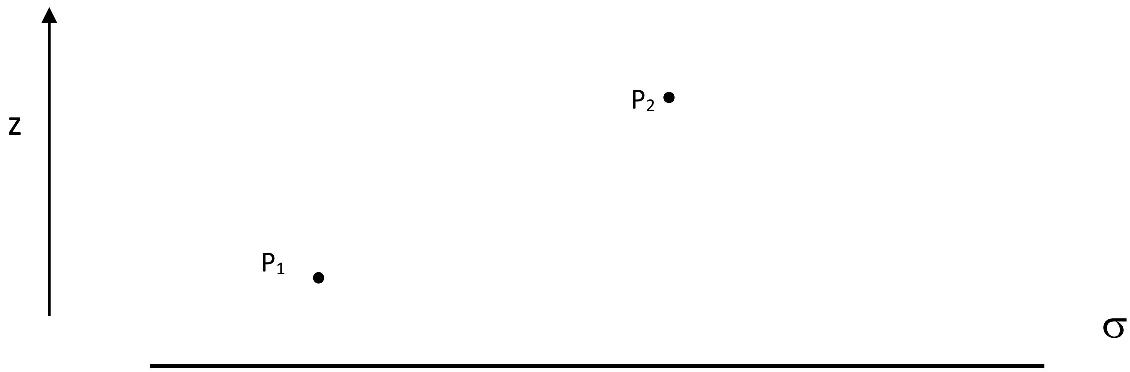
Preliminary Examination – Day 2

May, 2012

This test covers the topics of Electricity and Magnetism (Topic 1) and Quantum Mechanics (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work 2 problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

Electricity and Magnetism Group A. Answer only 2 Group A questions.

A1. Consider an E-field above an infinite plane at z = 0 with surface charge σ.



If σ is ***negative***, how much work is done in moving a positive charge from $\vec{P_1}$ (x_1, y_1, z_1), to $\vec{P_2}$ (x_2, y_2 z_2)? SHOW WORK!

A2. A current sheet (rectangular ribbon) in the x-y plane, with a surface current density $\vec{K} = K_0\hat{y}$ has dimensions ***a*** along the x axis and ***b*** along the y axis. What is the current **I**?

A3. A sphere of radius R contains a total charge Q distributed uniformly throughout its volume. It rotates about a diameter with constant angular speed ω. Assuming that the charge is not altered by the rotation

(a) find the volume current density at any point (r,θ,ϕ) within the sphere.

(b) find the volume current density at a point along the sphere's equator.

A4. In the classic image problem of a point charge, *q*, a distance *d* above a grounded infinite plane, a surface charge is induced on the plane.

(a) To determine the resulting E field above the grounded conductor you need to place what image charge where?

(b) How would the resulting E field above the conductor be changed if you moved a point charge $Q = -2q$ to the position $-d$ (below the plane)? Explain.

Electricity and Magnetism Group B. Answer only 2 Group B questions.

B1. The electric field $\vec{\boldsymbol{E}}$ of a point dipole at the origin with no special orientation with respect to the axes is given by (in spherical coordinates)

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r^3}[3(\vec{p}\bullet\hat{r})\hat{r} - \vec{p}].$$

Consider a point dipole $\overrightarrow{\boldsymbol{p_1}}$ fixed at the origin and oriented along the z axis (it cannot rotate or translate). Another point dipole $\overrightarrow{\boldsymbol{p_2}}$ is positioned along the z axis at a positive distance z_0 from the origin and oriented at an angle θ to the z axis, and in the same plane as $\overrightarrow{\boldsymbol{p_1}}$**.**

(a) What is the E-field at the position of $\overrightarrow{\boldsymbol{p_2}}$ due to $\overrightarrow{\boldsymbol{p_1}}$?

(b) What is the force on $\overrightarrow{\boldsymbol{p_2}}$ due to $\overrightarrow{\boldsymbol{p_1}}$?

(c) What is the torque on $\overrightarrow{\boldsymbol{p_2}}$?

(d) If the dipole of $\overrightarrow{\boldsymbol{p_2}}$ is free to rotate but not to translate, what final orientation does it tend to assume? Why?

B2. Consider an infinite slab of thickness 2*a* in the xy plane that extends to +*a* and –*a* along the z axis. The slab has a uniform polarization $\vec{\boldsymbol{P}}$ pointing along the z direction. This is ***NOT*** a Linear Isotropic Homogeneous (LIH) material.

(a) Calculate all the bound charge densities (ρ_b,σ_b).

(b) Calculate the $\vec{E}$ field above, below, and in the slab. Hint: You may find it useful to use the results for the E field of an infinite plane with appropriate surface charge density.

(c) Show that the boundary conditions are satisfied at both boundaries. Explain the direction of the normal at each surface and be careful of your SIGNS.

Note: Problems B3 and B4 are considered using a cylindrical coordinate system in which "s" is the radial coordinate.

B3. An infinitely long cylinder has a circular cross section of radius R. Current flows along its long axis, with a volume current density given by $\vec{J}(\vec{s}) = J_0 \frac{s}{R} \hat{z}$.

(a) Calculate the $\vec{B}$ field inside the cylinder.

(b) Calculate the vector potential $\vec{A}$ inside the cylinder. (Hint: Choose your zero for the vector potential at some radial distance s_0 outside the cylinder.)

B4. A current I flows down a long straight wire of radius a. The wire is made of linear magnetic material of susceptibility χ_m. The current is uniformly distributed.

(a) What are the $\vec{\boldsymbol{H}}$ and $\vec{\boldsymbol{B}}$ fields inside the wire?

(b) What are the $\vec{\boldsymbol{H}}$ and $\vec{\boldsymbol{B}}$ fields outside the wire?

(c) Show that the boundary conditions on $\vec{B}$ are satisfied at s = a.

Quantum Mechanics Group A. Answer only 2 Group A questions.

A1. A particle is in the rotational state $|\psi\rangle = C\left[3|2,1\rangle - 4i|4,-4\rangle\right]$ (C is a positive, real-valued number, and we write $|\ell, m\rangle$ for rotational eigenstates.)

(a) Find the value of C that normalizes $|\psi\rangle$.

We measure the z-component of its angular momentum, finding $L_z = -4\hbar$.

(b) What was the probability of obtaining this result?

Immediately after this first measurement we determine the quantity $\vec{L}^2$.

(c) Which results can be expected, and with what probability?

A2. An electron moves through free space with kinetic energy 5.0 keV. For what speed is the de Broglie wavelength of a free neutron equal to that of this electron?

A3. The eigenenergies for a particle in a one-dimensional infinite potential well of size a are given by

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}, \quad n = 1, 2, \ldots$$

Explain why the state $n = 0$ is impossible, from the point of view of

(a) the uncertainty principle.

(b) the particle's wave function.

A4. For a particle of mass m in a three-dimensional box of size a, write down expressions for the first three energy levels and give the degree of degeneracy for each of them.

Quantum Mechanics Group B. Answer only 2 Group B questions.

B1. A point particle with mass m moves in one dimension (x) in the potential $V(x)$. Its Hamiltonian is

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) \ .$$

(a) Is $[\hat{H}, \hat{x}]$ Hermitian? Show the work you did to answer this question.

(b) Show explicitly that $[\hat{H}, \hat{x}] = -i\frac{\hbar}{m}\hat{p}$.

B2. We consider the diatomic fluorine molecule $^{35}F_2$ (^{35}F = fluorine atom with mass $m \approx 35 m_{\text{proton}}$; the internuclear distance of the molecule $d = 1.42$ Å). The molecule rotates freely.

(a) Calculate the molecule's moment of inertia.

(b) Calculate the rotational constant B, including its SI units.

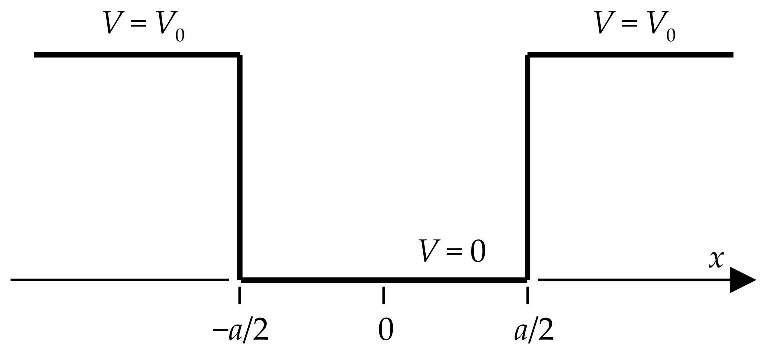
The molecule makes a rotational transition from $\ell = 8$ to $\ell = 2$, emitting a single photon.

(c) Calculate the energy of this photon in meV.

(d) Calculate the frequency (in Hz) of the photon. Note: If you couldn't find the photon energy in the previous part, you may assume it equals 20 meV – which is *not* the correct answer.

B3. Consider a particle in the well shown in the figure,

$V(x) = 0, |x| < a/2$, and
$V(x) = V_0,\ |x| > a/2$



(a) Assuming that $E < V_0$, write down the wave function in the regions $|x| < a/2$ and $|x| > a/2$. Consider separately even and odd solutions.

(b) Write down the matching equations and work out the equation allowing you to solve for the energy eigenvalues. Prove that for odd solutions this equation has the form

$$k\cot(ka/2) = -\kappa, \quad \text{where} \quad k = \sqrt{2mE}/\hbar \quad \text{and} \quad \kappa = \sqrt{2m(V_0 - E)}/\hbar$$

and obtain a similar equation for the even solutions.

B4. A particle in the harmonic oscillator potential, $V(x) = m\omega^2 x^2/2$, has as its initial wave function an even mixture of the first two normalized stationary states:

$$\Psi(x,0) = A[\psi_1(x) + \psi_2(x)].$$

(a) Find the A which normalizes Ψ.

At $t > 0$ the expectation value of the position x and momentum p have the form $\langle x \rangle = Bf(t),\ \langle p \rangle = Cg(t)$ where B, C are constants, and $f(t)$, $g(t)$ are functions of time.

(b) Find $f(t)$ and $g(t)$. (You don't have to find the constants B and C).

Physical Constants

speed of light .... $c = 2.998 \times 10^{8}$ m/s

Planck's constant .... $h = 6.626 \times 10^{-34}$ J·s

Planck's constant / 2π ... $\hbar = 1.055 \times 10^{-34}$ J·s

elementary charge .... $e = 1.602 \times 10^{-19}$ C

electric permittivity ...... $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m

magnetic permeability ... $\mu_0 = 1.257 \times 10^{-6}$ H/m

electrostatic constant ... $k = (4\pi\varepsilon_0)^{-1} = 8.988 \times 10^{9}$ m/F

electron mass .... $m_{\mathrm{el}} = 9.109 \times 10^{-31}$ kg

proton mass .... $m_{\mathrm{p}} = 1.673 \times 10^{-27}$ kg

1 bohr .... $a_0 = 0.5292$ Å

1 hartree .... $E_{\mathrm{h}} = 27.21$ eV

EQUATIONS THAT MAY BE HELPFUL

ELECTROSTATICS:

$$\oiint_A \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{\mathrm{enc}}}{\varepsilon_0} \qquad \vec{\nabla} \cdot \vec{\mathbf{E}} = \frac{\rho}{\varepsilon_0} \qquad \vec{\nabla} \times \vec{\mathbf{E}} = \vec{\mathbf{0}} \qquad \vec{\mathbf{E}} = -\vec{\nabla} V$$

$$-\int_{r_1}^{r_2} \vec{\mathbf{E}} \cdot d\vec{\ell} = V(r_2) - V(r_1) \qquad V(\vec{\mathbf{r}}) = \frac{1}{4\pi\varepsilon_0} \frac{q(\vec{\mathbf{r}}')}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|}$$

$$\text{work done} \quad W = -\int_{\vec{\mathbf{a}}}^{\vec{\mathbf{b}}} q\vec{\mathbf{E}} \cdot d\vec{\ell} = q\left[V(\vec{\mathbf{b}}) - V(\vec{\mathbf{a}})\right]$$

Multipole expansion:

$$V(\vec{\mathbf{r}}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r}\int \rho(\vec{\mathbf{r}}')d\tau' + \frac{1}{r^2}\int r'\cos(\theta')\rho(\vec{\mathbf{r}}')d\tau' + \frac{1}{r^3}\int (r')^2\left\{\tfrac{3}{2}\cos(\theta') - \tfrac{1}{2}\right\}\rho(\vec{\mathbf{r}}')d\tau' + \ldots\right]$$

where the first term is the monopole term, the second is the dipole term, the third is the quadrupole term..; $\vec{\mathbf{r}}$ and $\vec{\mathbf{r}}'$ are the field point and source point as usual and θ' is the angle between them.

$$\vec{\mathbf{D}} = \varepsilon_0\vec{\mathbf{E}} + \vec{\mathbf{P}} \qquad \vec{\nabla} \cdot \vec{\mathbf{D}} = \rho_{\mathrm{f}} \qquad \rho_b = -\vec{\nabla} \cdot \vec{\mathbf{P}} \qquad \sigma_b = \hat{\mathbf{P}} \cdot \hat{\mathbf{n}}$$

The above are true for ALL dielectrics. Confining ourselves to LIH dielectrics, we also have:

$\vec{\mathbf{D}} = \varepsilon \vec{\mathbf{E}}$ $\vec{\mathbf{P}} = \chi_e \varepsilon_0 \vec{\mathbf{E}}$ $\varepsilon = \varepsilon_0 (1 + \chi_e)$ $\kappa_e = \varepsilon / \varepsilon_0$ $\chi_e = \kappa_e - 1$

$C(\text{dielectric}) = \kappa_e C(\text{vacuum})$

Boundary Conditions:

$E_{2t} - E_{1t} = 0$ $E_{2n} - E_{1n} = \dfrac{\sigma}{\varepsilon_0}$

MAGNETOSTATICS:

Lorentz Force $\vec{\mathbf{F}} = q(\vec{\mathbf{v}} \times \vec{\mathbf{B}}) + q\vec{\mathbf{E}}$ Current densities $I = \int \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}}$ $\vec{I} = \int \vec{K} \bullet d\vec{l}$

Biot Savart Law $\vec{\mathbf{B}}(\vec{\mathbf{r}}) = \dfrac{\mu_0}{4\pi} I \int \dfrac{d\vec{\ell}' \times \hat{\Re}}{\Re^2}$ where R is the is the vector from the source point r′ to the field point r.

This can also be written (for surface currents) as $\vec{\mathbf{B}}(\vec{\mathbf{r}}) = \dfrac{\mu_0}{4\pi} \int \dfrac{\vec{\mathbf{K}} \times \hat{\Re}}{\Re^2} \cdot d\vec{\mathbf{A}}'$

For a straight wire segment $B = \dfrac{\mu_0 I}{4\pi s}\left[\sin\theta_2 - \sin\theta_1\right]$ where s is perpendicular distance from the wire.

For a circular loop of radius R, the B-field at a point on the axis is given by

$$B = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z^2)^{3/2}}$$

For an infinitely long solenoid, the B-field inside is given by $B = \mu_0 NI$ where N is the number of turns per unit length.

$\vec{\nabla} \cdot \vec{\mathbf{B}} = 0$ $\vec{\nabla} \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}}$ Ampere's law $\oint \vec{B} \bullet d\vec{l} = \mu_0 I_{enclosed}$

Magnetic vector potential A

$\vec{\mathbf{B}} = \vec{\nabla} \times \vec{\mathbf{A}}$

$\nabla^2 \vec{\mathbf{A}} = -\mu_0 \vec{\mathbf{J}} \quad \Rightarrow \quad \vec{\mathbf{A}} = \dfrac{\mu_0}{4\pi} \int \dfrac{\vec{\mathbf{J}}}{r - r'} d\tau'$

Also for line and surface currents $\vec{A} = \dfrac{\mu_0}{4\pi} \int \dfrac{I}{r - r'} d\vec{l}'$ $\vec{A} = \dfrac{\mu_0}{4\pi} \int \dfrac{\vec{K}}{r - r'} da'$

From Stokes theorem $\oint \vec{A} \bullet d\vec{l} = \int \vec{B}.d\vec{a}$

For a magnetic dipole m , $\vec{A}_{dipole} = \dfrac{\mu_0}{4\pi} \dfrac{\vec{m} \times \hat{r}}{r^2}$

Magnetic dipoles

Magnetic dipole moment of a current distribution is given by **m**=I∫d**a**

Torque on a magnetic dipole in a magnetic field $\vec{\boldsymbol{\tau}} = \vec{\mathbf{m}} \times \vec{\mathbf{B}}$

Force on a magnetic dipole $\vec{\mathbf{F}} = \vec{\nabla}(\vec{\mathbf{m}} \cdot \vec{\mathbf{B}})$ (minus sign?)

B field of a magnetic dipole $\vec{\mathbf{B}}(\vec{\mathbf{r}}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\vec{\mathbf{m}} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\right]$ (arrow over 2nd "m"?)

Dipole-dipole interaction energy is given by

$$U_{\mathrm{DD}} = \frac{\mu_0}{4\pi R^3}\left[(\vec{\mathbf{m}}_1 \cdot \vec{\mathbf{m}}_2) - 3(\vec{\mathbf{m}}_1 \cdot \hat{\mathbf{R}})(\vec{\mathbf{m}}_2 \cdot \hat{\mathbf{R}})\right] \quad \text{where } \vec{\mathbf{R}} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$$

Material with magnetization M produces a magnetic field equivalent to that of (bound)volume and surface current densities $\vec{\mathbf{J}}_b = \vec{\nabla} \times \vec{\mathbf{M}}$ and $\vec{\mathbf{K}}_b = \vec{\mathbf{M}} \times \hat{\mathbf{n}}$.

$$\oint \vec{\mathbf{H}} \cdot d\vec{\ell} = I_{\text{free, enclosed}} \qquad \vec{\mathbf{H}} = \frac{\vec{\mathbf{B}}}{\mu_0} - \vec{\mathbf{M}}$$

For linear magnetic material

$\mathbf{M}=\chi_m\mathbf{H}$ and $\mathbf{B}=\mu_0(1+\chi_m)\mathbf{H}$ **or** $\mathbf{B}=\mu\mathbf{H}$

Boundary Conditions: $B_{2n} - B_{1n} = 0 \qquad B_{2\parallel} - B_{1\parallel} = \mu_0 K$

Maxwell's Equations in vacuum:

1. $\vec{\nabla} \cdot \vec{\mathbf{E}} = \frac{\rho}{\varepsilon_0}$ Gauss' Law
2. $\vec{\nabla} \cdot \vec{\mathbf{B}} = 0$
3. $\vec{\nabla} \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$ Faraday's Law
4. $\vec{\nabla} \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}} + \varepsilon_0 \mu_0 \frac{\partial \vec{\mathbf{E}}}{\partial t}$ Ampere's Law with Maxwell's correction

Maxwell's Equations in LIH media:

1. $\vec{\nabla}\cdot\vec{\mathbf{E}} = \frac{\rho_f}{\varepsilon}$ Gauss' Law

2. $\vec{\nabla}\cdot\vec{\mathbf{B}} = 0$

3. $\vec{\nabla}\times\vec{\mathbf{E}} = -\frac{\partial\vec{\mathbf{B}}}{\partial t}$ Faraday's Law

4. $\vec{\nabla}\times\vec{\mathbf{B}} = \mu\vec{\mathbf{J}}_f + \varepsilon\mu\frac{\partial\vec{\mathbf{E}}}{\partial t}$ Ampere's Law with Maxwell's correction

Alternative ways of writing Faraday's Law: $\oint\vec{\mathbf{E}}\cdot d\vec{\ell} = -\frac{d\Phi}{dt}$

Mutual and self inductance: $\Phi_2 = M_{21}I_1$ and $M_{21} = M_{12}$ $\Phi = LI$

Energy stored in a magnetic field: $W = \frac{1}{2\varepsilon_0}\int_V B^2 d\tau = \frac{1}{2}LI^2 = \frac{1}{2}\oint\vec{\mathbf{A}}\cdot\vec{\mathbf{I}}d\ell$

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl : $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

- *Schrödinger Equation.* General and time-independent:

$$
\begin{aligned}
i\hbar\frac{\partial\Psi}{\partial t} &= \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\Psi \\
H\psi &= E\psi \\
\Psi &= \psi e^{-iEt/\hbar}
\end{aligned} \tag{1}
$$

- *Formalism and Operator Algebra.* This is probably actually the easiest topic for people who have taken 143a. Some equations of use are:

$$
\begin{aligned}
\langle\alpha|\beta\rangle &= \langle\beta|\alpha\rangle^* \\
\hat{O} &= \hat{O}^\dagger \text{ if } \hat{O} \text{ is Hermitian} \\
\langle\alpha|\hat{O}|\beta\rangle &= \beta\langle\alpha|\beta\rangle \text{ if } |\beta\rangle \text{ is an eigenstate of } \hat{O} \\
&= \int_{-\infty}^{\infty} \mathrm{d}x\, \alpha^*(x)\hat{O}(x)\beta(x) \text{ in 1D} \\
&= A^\dagger \times O \times B \text{ as matrices} \\
[\hat{A},\hat{B}] &= \hat{A}\hat{B} - \hat{B}\hat{A} \\
\therefore\ [\hat{A},\hat{B}] &= -[\hat{B},\hat{A}] \\
[\hat{A}\hat{B},\hat{C}] &= [\hat{A},\hat{C}]\hat{B} + \hat{A}[\hat{B},\hat{C}]
\end{aligned} \tag{2}
$$

A Hermitian matrix has real eigenvalues; observables thus must be represented by Hermitian operators.

- *Position and momentum.*

$$
\begin{aligned}
\hat{p} &= -i\hbar\nabla \\
[\hat{x},\hat{p}] &= i\hbar \\
[\hat{f}(x),\hat{p}] &= i\hbar\frac{\mathrm{d}f}{\mathrm{d}x}
\end{aligned} \tag{3}
$$

$$\psi(x) = \langle x|\psi\rangle = \int_{-\infty}^{\infty} dp\, \langle x|p\rangle\langle p|\psi\rangle = \int_{-\infty}^{\infty} dp\, \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}\,\varphi(p)$$

$E_\ell = B\ell(\ell+1)$, with $\ell = 0,1,2,\ldots$