UNL Department of Physics and Astronomy

Preliminary Examination – Day 1

May, 2012

This test covers the topics of Mechanics (Topic 1) and Thermodynamics and Statistical Mechanics (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work 2 problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

Mechanics Group A. Answer only 2 Group A questions.

A1. A one-dimensional potential is given by $V(x) = (1 - e^{-ax^2})$ joules, where a = 5 m^{-2}. A point mass of 2 kg executes small-amplitude motion about the bottom of this potential well. Calculate the period of this motion.

A2. Particles of mud are thrown from the rim of a wheel that is rolling without slipping. If the forward speed of the wheel is v and its radius is R, find the maximum height above the ground which the mud reaches. Ignore air drag and fenders.

A3. A plank of weight W and length √3*R lies in a smooth circular trough of radius R. At one end of the plank (attached to it) is a small mass of weight W/2. When the plank is in equilibrium, what angle does it make with the horizontal?

A4. Consider the force function

$$\vec{\boldsymbol{F}} = y\hat{\boldsymbol{x}} + x\hat{\boldsymbol{y}} + z^2\hat{\boldsymbol{z}}\ .$$

a) Is it a conservative function?

b) If the answer to a) is "yes,"find the corresponding potential function.

c) Calculate the line integral

$$\int \vec{F} \cdot d\vec{r}$$

for the path $\vec{\boldsymbol{r}} = t\hat{\boldsymbol{x}} + t^2\hat{\boldsymbol{y}} + t\hat{\boldsymbol{z}}$ from (0,0,0) to (3,9,3).

Mechanics Group B. Answer only 2 Group B questions.

B1. The interaction between an atom and an ion at distances greater than contact is given by the potential energy $V(r) = -C\,r^{-4}$.

(a) Sketch the effective potential energy as a function of r, assuming some value of the angular momentum ℓ.

(b) If the total energy of the ion exceeds the maximum value of the effective potential energy V_o, the ion can strike the atom (i.e. reach r = 0). Find V_o in terms of the angular momentum ℓ.

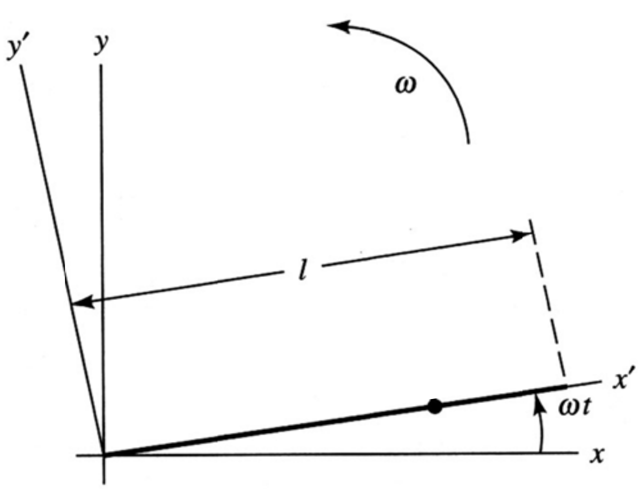
(c) If the ion starts out very far away from the atom with velocity v_o, what is the largest possible value of the impact parameter b such that the ion will strike the atom?

B2. A smooth rod of length l rotates in a plane with a constant angular speed ω about an axis fixed at the end of the rod and perpendicular to the plane of rotation. A bead of mass m is initially released from at rest (relative to the rod) at its midpoint. The bead is constrained to move along the rod without friction. Calculate:

(a) the displacement of the bead along the rod as a function of time.

(b) the time when the bead leaves the end of rod.

(c) the linear speed (relative to the rod) with which the bead leaves the end of rod.

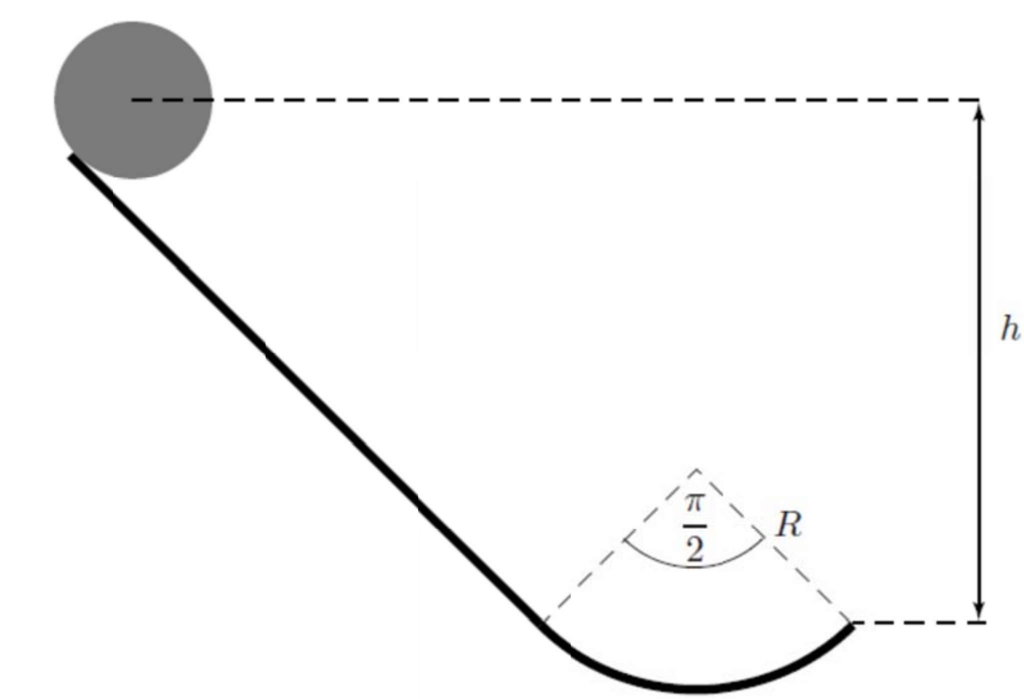


B3. A sphere of radius r and mass m rolls without slipping down a ramp inclined at 45° to the horizontal. At the bottom of the ramp is a quarter-circle arc of radius R that meets the ramp smoothly (see figure). When the sphere is released, its center is a distance h above the right end of the circular track.

(a) What is the velocity of the sphere when it reaches the end of the circular track?

(b) What is the maximum height reached by the sphere after it loses contact with the track?

(c) What is the horizontal distance of the sphere from the end of the track at the time it reaches its maximum height in (b)?

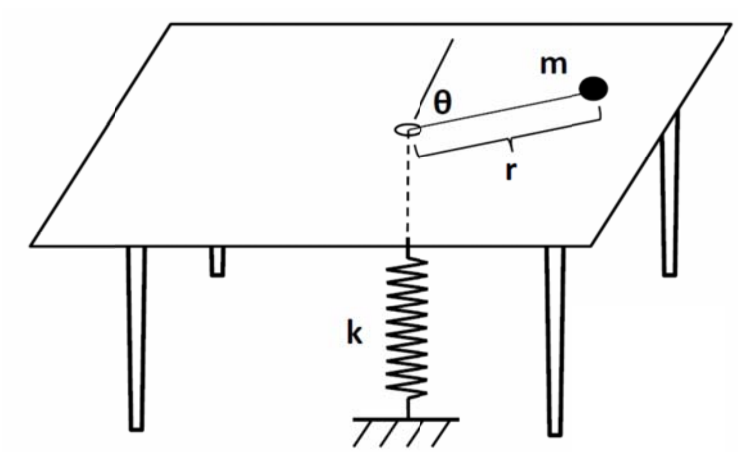


B4. Consider a small mass moving on a frictionless table attached to a vertical massless spring of spring constant k, as shown, by a massless string. The spring exerts no force when the mass is directly above it. At t = 0, the mass is given a velocity v_0 perpendicular to the string of which a part of length a lies on the table at that time.

(a) Write down the Lagrangian function for the system in terms of the generalized coordinates r = the length of the string on the table, and θ, the angle between the string on the table and some arbitrary radial line painted on the table.

(b) Is either coordinate ignorable? Which, if any?

(c) Write down the Lagrangian equations of motion.



Thermodynamics and Statistics Group A. Answer only 2 Group A questions.

A1. An ideal monatomic gas expands isobarically at 100 kPa from a volume of 1 liter to a volume of 2 liters. What is the amount of heat that the gas absorbs in the process?

A2. An isolated container is split by a wall into two compartments. The first compartment, of volume V_1, contains N molecules of an ideal gas. The other compartment of volume V_2 is evacuated. The wall is removed, the gas fills the whole container, and after some time the system reaches equilibrium. Compute the change in the Helmholtz free energy between the initial and the final equilibrium states.

A3. A 1.5 kg glass brick is heated to 180°C and then plunged into a cold bath containing 10 kg of water at 20°C. Assume that none of the water boils and that there is no heating of the surroundings. What is the final temperature of the water and the glass? The specific heat of glass is approximately 750 J/kg K, and that of water 4184 J/kg K. It takes 334 J to melt 1 gram of ice.

A4. What is the minimum amount of electric energy that a household freezer needs to consume in order to freeze 1 kg of water, assuming that the room temperature is 20°C? The heat capacity of water is 4184 J/kg K.

Thermodynamics and Statistics Group B. Answer only 2 Group B questions.

B1. One mole of an ideal monatomic gas undergoes a process starting from equilibrium at the pressure of 100 kPa and temperature of 100°C. In the final equilibrium state the pressure is 200 kPa and the temperature is 50°C. Find the change in the entropy of the gas in this process assuming that

(a) the process is reversible.

(b) the process is irreversible.

B2. Two dice are rolled and the sum S on their top faces is examined.

(a) List the possible outcomes and their respective probabilities for S.

(b) What is the most probable value of S, and which combination(s) of (die 1, die 2) corresponds to it?

(c) Find the mean value of S and its standard deviation for the case when the double dice throw is made a large number of times.

B3. Show that the heat absorbed by a system in an isobaric process is equal to the change of its enthalpy, under the condition that only mechanical work is involved.

B4. Derive an equation connecting the volume and temperature of a fixed amount of monatomic ideal gas undergoing a reversible adiabatic process.

EQUATIONS THAT MAY BE HELPFUL

General efficiency η of a heat engine producing work $|W|$ while taking in heat $|Q_h|$ is $\eta = \frac{|W|}{|Q_h|}$.

For a Carnot cycle operating as a heat engine between reservoirs at T_h and at T_c the efficiency becomes $\eta_C = \frac{T_h - T_c}{T_h}$.

Clausius' theorem: $\sum_{i=1}^{N} \frac{Q_i}{T_i} \leq 0$ which become $\sum_{i=1}^{N} \frac{Q_i}{T_i} = 0$ for a reversible cyclic process of N steps.

$\frac{dP}{dT} = \lambda/(T\Delta V)$; specific heat of water: 4186 J/(kg*K); Latent heat of ice melting: 334 J/g

$\overrightarrow{\boldsymbol{a'}} = \vec{\boldsymbol{a}} - 2\vec{\boldsymbol{\omega}} \times \overrightarrow{\boldsymbol{v'}} - \dot{\vec{\boldsymbol{\omega}}} \times \overrightarrow{\boldsymbol{r'}} - \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \overrightarrow{\boldsymbol{r'}})$

H = E + PV F = E – TS G = F + PV Ω = F – μN

dE = TdS – PdV + μdN dS = dE/T + PdV/T – μdN/T dH = TdS + VdP + μdN

dF = -SdT – PdV + μdN dG = -SdT + VdP + μdN dΩ = -SdT – PdV - Ndμ

UNL Department of Physics and Astronomy

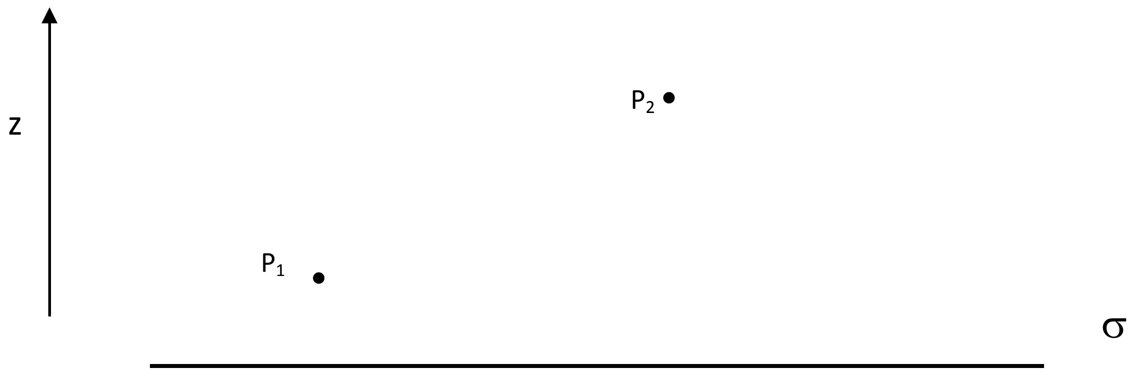
Preliminary Examination – Day 2

May, 2012

This test covers the topics of Electricity and Magnetism (Topic 1) and Quantum Mechanics (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work 2 problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

Electricity and Magnetism Group A. Answer only 2 Group A questions.

A1. Consider an E-field above an infinite plane at z = 0 with surface charge σ.



If σ is ***negative***, how much work is done in moving a positive charge from $\vec{P_1}$ (x_1, y_1, z_1), to $\vec{P_2}$ (x_2, y_2 z_2)? SHOW WORK!

A2. A current sheet (rectangular ribbon) in the x-y plane, with a surface current density $\vec{K} = K_0\hat{y}$ has dimensions ***a*** along the x axis and ***b*** along the y axis. What is the current **I**?

A3. A sphere of radius R contains a total charge Q distributed uniformly throughout its volume. It rotates about a diameter with constant angular speed ω. Assuming that the charge is not altered by the rotation

(a) find the volume current density at any point (r,θ,ϕ) within the sphere.

(b) find the volume current density at a point along the sphere's equator.

A4. In the classic image problem of a point charge, *q*, a distance *d* above a grounded infinite plane, a surface charge is induced on the plane.

(a) To determine the resulting E field above the grounded conductor you need to place what image charge where?

(b) How would the resulting E field above the conductor be changed if you moved a point charge $Q = -2q$ to the position -*d* (below the plane)? Explain.

Electricity and Magnetism Group B. Answer only 2 Group B questions.

B1. The electric field $\vec{\boldsymbol{E}}$ of a point dipole at the origin with no special orientation with respect to the axes is given by (in spherical coordinates)

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r^3}[3(\vec{p} \bullet \hat{r})\hat{r} - \vec{p}].$$

Consider a point dipole $\overrightarrow{\boldsymbol{p_1}}$ fixed at the origin and oriented along the z axis (it cannot rotate or translate). Another point dipole $\overrightarrow{\boldsymbol{p_2}}$ is positioned along the z axis at a positive distance z_0 from the origin and oriented at an angle θ to the z axis, and in the same plane as $\overrightarrow{\boldsymbol{p_1}}$.

(a) What is the E-field at the position of $\overrightarrow{\boldsymbol{p_2}}$ due to $\overrightarrow{\boldsymbol{p_1}}$?

(b) What is the force on $\overrightarrow{\boldsymbol{p_2}}$ due to $\overrightarrow{\boldsymbol{p_1}}$?

(c) What is the torque on $\overrightarrow{\boldsymbol{p_2}}$?

(d) If the dipole of $\overrightarrow{\boldsymbol{p_2}}$ is free to rotate but not to translate, what final orientation does it tend to assume? Why?

B2. Consider an infinite slab of thickness 2*a* in the xy plane that extends to +*a* and –*a* along the z axis. The slab has a uniform polarization $\vec{\boldsymbol{P}}$ pointing along the z direction. This is ***NOT*** a Linear Isotropic Homogeneous (LIH) material.

(a) Calculate all the bound charge densities (ρ_b,σ_b).

(b) Calculate the $\vec{E}$ field above, below, and in the slab. Hint: You may find it useful to use the results for the E field of an infinite plane with appropriate surface charge density.

(c) Show that the boundary conditions are satisfied at both boundaries. Explain the direction of the normal at each surface and be careful of your SIGNS.

Note: Problems B3 and B4 are considered using a cylindrical coordinate system in which "s" is the radial coordinate.

B3. An infinitely long cylinder has a circular cross section of radius R. Current flows along its long axis, with a volume current density given by $\vec{J}(\vec{s}) = J_0 \frac{s}{R}\hat{z}$.

(a) Calculate the $\vec{B}$ field inside the cylinder.

(b) Calculate the vector potential $\vec{A}$ inside the cylinder. (Hint: Choose your zero for the vector potential at some radial distance s_0 outside the cylinder.)

B4. A current I flows down a long straight wire of radius a. The wire is made of linear magnetic material of susceptibility χ_m. The current is uniformly distributed.

(a) What are the $\vec{\boldsymbol{H}}$ and $\vec{\boldsymbol{B}}$ fields inside the wire?

(b) What are the $\vec{\boldsymbol{H}}$ and $\vec{\boldsymbol{B}}$ fields outside the wire?

(c) Show that the boundary conditions on $\vec{B}$ are satisfied at s = a.

Quantum Mechanics Group A. Answer only 2 Group A questions.

A1. A particle is in the rotational state $|\psi\rangle = C\left[3|2,1\rangle - 4i|4,-4\rangle\right]$ (C is a positive, real-valued number, and we write $|\ell, m\rangle$ for rotational eigenstates.)

(a) Find the value of C that normalizes $|\psi\rangle$.

We measure the z-component of its angular momentum, finding $L_z = -4\hbar$.

(b) What was the probability of obtaining this result?

Immediately after this first measurement we determine the quantity $\vec{L}^2$.

(c) Which results can be expected, and with what probability?

A2. An electron moves through free space with kinetic energy 5.0 keV. For what speed is the de Broglie wavelength of a free neutron equal to that of this electron?

A3. The eigenenergies for a particle in a one-dimensional infinite potential well of size a are given by

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}, \quad n = 1, 2, \ldots$$

Explain why the state $n = 0$ is impossible, from the point of view of

(a) the uncertainty principle.

(b) the particle's wave function.

A4. For a particle of mass m in a three-dimensional box of size a, write down expressions for the first three energy levels and give the degree of degeneracy for each of them.

Quantum Mechanics Group B. Answer only 2 Group B questions.

B1. A point particle with mass m moves in one dimension (x) in the potential $V(x)$. Its Hamiltonian is

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) \ .$$

(a) Is $[\hat{H}, \hat{x}]$ Hermitian? Show the work you did to answer this question.

(b) Show explicitly that $[\hat{H}, \hat{x}] = -i\frac{\hbar}{m}\hat{p}$.

B2. We consider the diatomic fluorine molecule $^{35}F_2$ (^{35}F = fluorine atom with mass $m \approx 35 m_{\text{proton}}$; the internuclear distance of the molecule $d = 1.42$ Å). The molecule rotates freely.

(a) Calculate the molecule's moment of inertia.

(b) Calculate the rotational constant B, including its SI units.

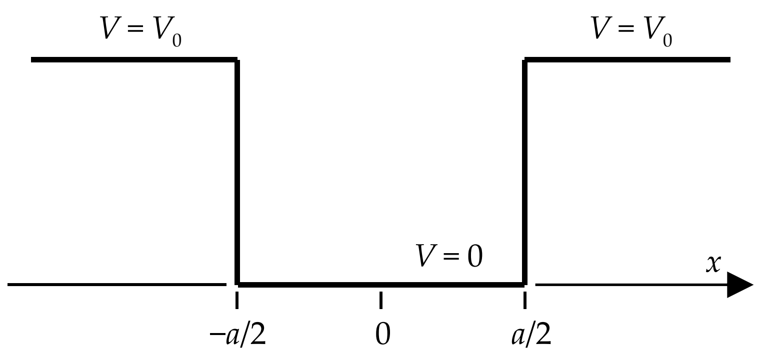
The molecule makes a rotational transition from $\ell = 8$ to $\ell = 2$, emitting a single photon.

(c) Calculate the energy of this photon in meV.

(d) Calculate the frequency (in Hz) of the photon. Note: If you couldn't find the photon energy in the previous part, you may assume it equals 20 meV – which is *not* the correct answer.

B3. Consider a particle in the well shown in the figure,

$V(x) = 0, |x| < a/2$, and
$V(x) = V_0, \; |x| > a/2$



(a) Assuming that $E < V_0$, write down the wave function in the regions $|x| < a/2$ and $|x| > a/2$. Consider separately even and odd solutions.

(b) Write down the matching equations and work out the equation allowing you to solve for the energy eigenvalues. Prove that for odd solutions this equation has the form

$$k\cot(ka/2) = -\kappa, \quad \text{where} \quad k = \sqrt{2mE}/\hbar \quad \text{and} \quad \kappa = \sqrt{2m(V_0 - E)}/\hbar$$

and obtain a similar equation for the even solutions.

B4. A particle in the harmonic oscillator potential, $V(x) = m\omega^2 x^2/2$, has as its initial wave function an even mixture of the first two normalized stationary states:

$$\Psi(x,0) = A[\,\psi_1(x) + \psi_2(x)\,].$$

(a) Find the A which normalizes Ψ.

At $t > 0$ the expectation value of the position x and momentum p have the form $\langle x \rangle = Bf(t), \;\; \langle p \rangle = Cg(t)$ where B,C are constants, and $f(t)$, $g(t)$ are functions of time.

(b) Find $f(t)$ and $g(t)$. (You don't have to find the constants B and C).

Physical Constants

speed of light .... $c = 2.998 \times 10^{8}$ m/s

Planck's constant .... $h = 6.626 \times 10^{-34}$ J·s

Planck's constant / 2π ... $\hbar = 1.055 \times 10^{-34}$ J·s

elementary charge .... $e = 1.602 \times 10^{-19}$ C

electric permittivity ...... $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m

magnetic permeability ... $\mu_0 = 1.257 \times 10^{-6}$ H/m

electrostatic constant ... $k = (4\pi\varepsilon_0)^{-1} = 8.988 \times 10^{9}$ m/F

electron mass .... $m_{\mathrm{el}} = 9.109 \times 10^{-31}$ kg

proton mass .... $m_{\mathrm{p}} = 1.673 \times 10^{-27}$ kg

1 bohr .... $a_0 = 0.5292$ Å

1 hartree .... $E_{\mathrm{h}} = 27.21$ eV

EQUATIONS THAT MAY BE HELPFUL

Electrostatics:

$$\oiint_A \vec{E} \bullet d\vec{A} = \frac{q_{enc}}{\varepsilon_0} \qquad \vec{\nabla} \bullet \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{\nabla} X \vec{E} = 0 \qquad \vec{E} = -\nabla V$$

$$-\int_{r_1}^{r_2} \vec{E} \bullet d\vec{l} = V(r_2) - V(r_1) \qquad V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q(\vec{r}^{\,'})}{\left|\vec{r} - \vec{r}^{\,'}\right|}$$

Work done $W = -\int_{\vec{a}}^{\vec{b}} q\vec{E}.d\vec{l} = q[V(\vec{b}) - V(\vec{a})]$

Multipole expansion:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r}\int \rho(\vec{r}^{\,'})d\tau^{'} + \frac{1}{r^2}\int r^{'}Cos\theta^{'}\rho(\vec{r}^{\,'})d\tau^{'} + \frac{1}{r^3}\int (r^{'})^2(\frac{3}{2}Cos\theta^{'} - \frac{1}{2})\rho(\vec{r}^{\,'})d\tau^{'} + ...\right]$$

where the first term is the monopole term, the second is the dipole term, the third is the quadrupole term..; r and r' are the field point and source point as usual and θ' is the angle between them.

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P} \qquad \vec{\nabla} \bullet \vec{D} = \rho_f \qquad \rho_b = -\vec{\nabla} \bullet \vec{P} \qquad \sigma_b = \vec{P} \bullet \hat{n}$$

The above are true for ALL dielectrics. Confining ourselves to LIH dielectrics, we also have:

$$\vec{D} = \varepsilon\vec{E} \qquad \vec{P} = \chi_e\varepsilon_0\vec{E} \qquad \varepsilon = \varepsilon_0(1 + \chi_e) \qquad \kappa_e = \varepsilon/\varepsilon_0 \qquad \chi_e = \kappa_e - 1$$

C(dielectric)=κ_e C (vacuum)

Boundary Conditions:

$$E_{2t} - E_{1t} = 0 \qquad E_{2n} - E_{1n} = \frac{\sigma}{\varepsilon_0}$$

<u>Magnetostatics:</u>

Lorentz Force $\vec{F} = q(\vec{v}\times\vec{B}) + q\vec{E}$ Current densities $\vec{I} = \int \vec{J}\bullet d\vec{A}$ $\vec{I} = \int \vec{K}\bullet d\vec{l}$

Biot Savart Law $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l}'\times\hat{\Re}}{\Re^2}$ where R is the is the vector from the source point r' to the field point r.

This can also be written (for surface currents) as $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{K}\times\hat{\Re}}{\Re^2}\bullet dA'$

For a straight wire segment $B = \frac{\mu_0 I}{4\pi s}[Sin(\theta_2) - Sin(\theta_1)]$ where s is perpendicular distance from the wire.

For a circular loop of radius R, the B field at a point on the axis is given by

$$B = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}$$

For an infinitely long solenoid, the B field inside is given by B=μ_0NI where N is the number of turns per unit length.

$\vec{\nabla}\bullet\vec{B} = 0$ $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ Ampere's law $\oint \vec{B}\bullet d\vec{l} = \mu_0 I_{enclosed}$

<u>Magnetic vector potential A</u>

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$\nabla^2\vec{A} = -\mu_0\vec{J} \Rightarrow \vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}}{r-r'}d\tau'$$

Also for line and surface currents $\vec{A} = \frac{\mu_0}{4\pi}\int \frac{I}{r-r'}d\vec{l}'$ $\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{K}}{r-r'}da'$

From Stokes theorem $\oint \vec{A}\bullet d\vec{l} = \int \vec{B}.d\vec{a}$

For a magnetic dipole m , $\vec{A}_{dipole} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$

<u>Magnetic dipoles</u>

Magnetic dipole moment of a current distribution is given by **m**=I∫d**a**

Torque on a magnetic dipole in a magnetic field **τ=mXB**

Force on a magnetic dipole $\vec{F} = \nabla(\vec{m}\bullet\vec{B})$

B field of a magnetic dipole $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\vec{m}\bullet\hat{r})\hat{r} - \hat{m}]$

Dipole-dipole interaction energy is given by

$U_{DD} = \frac{\mu_0}{4\pi R^3}\left[(\vec{m}_1\bullet\vec{m}_2) - 3(\vec{m}_1\bullet\hat{R})(\vec{m}_2\bullet\hat{R})\right]$ where **R=r$_1$-r$_2$**

Material with magnetization M produces a magnetic field equivalent to that of (bound)volume and surface current densities $\vec{J}_b = \vec{\nabla} X \vec{M}$ and $\vec{K}_b = \vec{M} \times \hat{n}$.

$\oint \vec{H} \bullet d\vec{l} = I_{freeenclosed}$ $\qquad \vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$

For linear magnetic material

M=χ_m**H** and **B**=$\mu_0(1+\chi_m)$**H** **or B**=μ**H**

Boundary Conditions: B_{2n}-B_{1n}=0 $\qquad B_{2//}$-$B_{1//}$=μ_0K

Maxwell's Equations in vacuum:

1.$\vec{\nabla} \bullet \vec{E} = \frac{\rho}{\varepsilon_0}$ Gauss' Law

2.$\vec{\nabla} \bullet \vec{B} = 0$

3.$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ Faradays law

4. $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ Ampere's Law with Maxwell's correction

Maxwell's Equations in LIH media :

1.$\vec{\nabla} \bullet \vec{E} = \frac{\rho_f}{\varepsilon}$ Gauss' Law

2.$\vec{\nabla} \bullet \vec{B} = 0$

3.$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ Faradays law

4. $\vec{\nabla} \times \vec{B} = \mu \vec{J}_f + \mu \varepsilon \frac{\partial \vec{E}}{\partial t}$ Ampere's Law with Maxwell's correction

Alternative ways of writing Faraday's Law $\oint \vec{E} \bullet d\vec{l} = -\frac{d\phi}{dt}$

Mutual and self inductance: ϕ_2=$M_{21}I_1$ and M_{21}=M_{12}

ϕ=LI

Energy stored in a magnetic field: $W = \frac{1}{2\varepsilon_0} \int_V B^2 d\tau = \frac{1}{2} LI^2 = \frac{1}{2} \oint \vec{A} \bullet \vec{I} dl$

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl : $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

- *Schrödinger Equation.* General and time-independent:

$$\begin{aligned} i\hbar\frac{\partial \Psi}{\partial t} &= \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\Psi \\ H\psi &= E\psi \\ \Psi &= \psi e^{-iEt/\hbar} \end{aligned} \tag{1}$$

- *Formalism and Operator Algebra.* This is probably actually the easiest topic for people who have taken 143a. Some equations of use are:

$$\begin{aligned} \langle\alpha|\beta\rangle &= \langle\beta|\alpha\rangle^* \\ \hat{O} &= \hat{O}^\dagger \text{ if } \hat{O} \text{ is Hermitian} \\ \langle\alpha|\hat{O}|\beta\rangle &= \beta\langle\alpha|\beta\rangle \text{ if } |\beta\rangle \text{ is an eigenstate of } \hat{O} \\ &= \int_{-\infty}^{\infty} \mathrm{d}x\, \alpha^*(x)\hat{O}(x)\beta(x) \text{ in 1D} \\ &= A^\dagger \times O \times B \text{ as matrices} \\ [\hat{A},\hat{B}] &= \hat{A}\hat{B} - \hat{B}\hat{A} \\ \therefore\ [\hat{A},\hat{B}] &= -[\hat{B},\hat{A}] \\ [\hat{A}\hat{B},\hat{C}] &= [\hat{A},\hat{C}]\hat{B} + \hat{A}[\hat{B},\hat{C}] \end{aligned} \tag{2}$$

A Hermitian matrix has real eigenvalues; observables thus must be represented by Hermitian operators.

- *Position and momentum.*

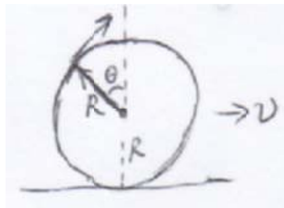
$$\begin{aligned} \hat{p} &= -i\hbar\nabla \\ [\hat{x},\hat{p}] &= i\hbar \\ [\hat{f}(x),\hat{p}] &= i\hbar\frac{\mathrm{d}f}{\mathrm{d}x} \end{aligned} \tag{3}$$

$$\psi(x) = \langle x|\psi\rangle = \int_{-\infty}^{\infty} dp\, \langle x|p\rangle\langle p|\psi\rangle = \int_{-\infty}^{\infty} dp\, \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}\,\varphi(p)$$

Mechanics Group A.
Solutions:

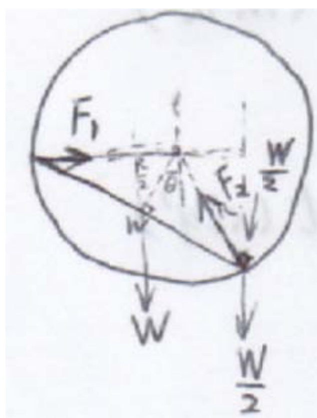
A1. $$\omega = \frac{2\pi}{T} = \sqrt{\frac{1}{m}\frac{d^2V}{dx^2}\bigg|_{x=0}} \quad ⇨ \quad T = 2\pi\sqrt{\frac{m}{\frac{d^2V}{dx^2}\big|_{x=0}}} = 2\pi\sqrt{\frac{m}{2a}} = 2.81\ s$$

A2.



$$h = R + R\cos\theta + \frac{(v\sin\theta)^2}{2g} = -\frac{v^2}{2g}\cos^2\theta + R + R\cos\theta + \frac{v^2}{2g}$$
$$⇨\ \cos\theta = \frac{Rg}{v^2},\quad |\cos\theta| \le 1 \quad ⇨ \quad h_{max} = R + \frac{v^2}{2g} + \frac{R^2 g}{2v^2}$$

A3.



Balance of torque:

$$W\frac{R}{2}\sin\theta = \frac{W}{2}R\sin\left(\frac{\pi}{3} - \theta\right)$$
$$⇨\ \theta = \frac{\pi}{6} = 30°$$

A4. (a) Yes.

(b) $\vec{\boldsymbol{F}} = -\boldsymbol{\nabla}V\ ⇨\ V = -xy - \frac{z^3}{3}$

(c) $\vec{\boldsymbol{r}} = \vec{\boldsymbol{x}} + \vec{\boldsymbol{y}} + \vec{\boldsymbol{z}} \quad ⇨ \quad x = t,\ y = t^2,\ z = t$

$$⇨\ \vec{\boldsymbol{F}} = t^2\hat{\boldsymbol{x}} + t\hat{\boldsymbol{y}} + t^2\hat{\boldsymbol{z}} \quad ⇨ \quad \int \vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{r}} = \int_{\boldsymbol{0}}^{\boldsymbol{3}} (t^2 + 2t^2 + t^2)dt = 36$$

Mechanics Group B.
Solutions:

B1. (a)

$$V_{eff}(r) = V(r) + \frac{l^2}{2mr^2} = \frac{l^2}{2mr^2} - \frac{C}{r^4}$$

(b) Let

$$\frac{dV_{eff}}{dr} = -\frac{l^2}{mr^3} + \frac{4C}{r^5} = 0 \quad ⇨ \quad r = \frac{2}{l}\sqrt{mC}$$

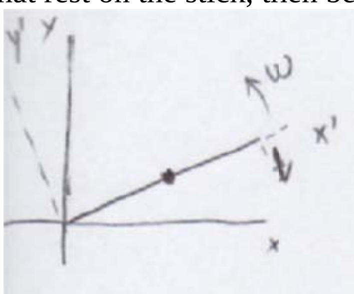
$$\left.\frac{d^2V_{eff}}{dr^2}\right|_{r=\frac{2}{l}\sqrt{mC}} = \left.\frac{3l^2}{mr^4} - \frac{20C}{r^6}\right|_{r=\frac{2}{l}\sqrt{mC}} = -\frac{l^6}{8m^3C^3} < 0$$

$$V_{eff}(r)_{max} = V_{eff}\left(r = \frac{2}{l}\sqrt{mC}\right) = \frac{l^4}{16m^2C} = V_0$$

(c)

$$E = \frac{1}{2}mv_0^2 \geq V_0, \quad l = mbv_0 \quad ⇨ \quad b^4 \leq \frac{8C}{mv_0^2} \quad ⇨ \quad b_{max} = \left(\frac{8C}{mv_0^2}\right)^{1/4}$$

B2. (a) We choose coordinate that rest on the stick, then bead (x', y')



$F' = ma' = mr\omega^2$ and $r = x' \geq \frac{l}{2}$

$$\frac{d^2x'}{dt^2} = x'\omega^2$$

$$x'(t=0) = \frac{l}{2}, \qquad \dot{x}'(t=0) = 0$$

$$⇨ x'^{(t)} = \frac{l}{4}(e^{\omega t} + e^{-\omega t}) = \frac{l}{2}arccosh(\omega t)$$

(b) For $x'(t) = L$, then time $t = \frac{1}{\omega}arcosh(2)$

(c) $v' = \dot{x}'\left(t = \frac{1}{\omega}arcosh(2)\right) = \frac{l\omega}{2}sinh(\omega t) = \frac{\sqrt{3}}{2}l\omega$

B3. (a) Momentum of inertia for this sphere $I = \frac{2}{5}mr^2$

Because of the conservation of energy, we have:

$$mg\left(h - \frac{r}{\sqrt{2}}\right) = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

With $v = r\omega$. Thus, we get

$$v = \sqrt{\frac{10}{7} g \left(h - \frac{r}{\sqrt{2}}\right)}$$

(b)

$$H_{max} = \frac{\left(v \sin\frac{\pi}{4}\right)^2}{2g} + \frac{r}{\sqrt{2}} = \frac{5}{14} h + \frac{9r}{14\sqrt{2}}$$

(c)

$$s = \left(v \cos\frac{\pi}{4}\right)\left(\frac{v \sin\frac{\pi}{4}}{g}\right) = \frac{5}{7}\left(h - \frac{r}{\sqrt{2}}\right)$$

B4. (a) $V(r) = \frac{1}{2} kr^2,\ \ l = mav_0$ Angular momenta

$$L = T - V = \frac{1}{2} m\dot{r}^2 + \frac{l^2}{2mr^2} - \frac{1}{2} kr^2 = \frac{1}{2} m\dot{r}^2 + \frac{ma^2 v_0^2}{2r^2} - \frac{1}{2} kr^2$$

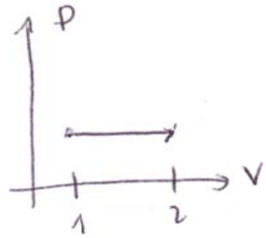
(b) Yes, angle θ. Potential is independent of angle θ, central force field.

(c)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0 \quad \Rightarrow \quad m\ddot{r} + kr - \frac{ma^2 v_0^2}{r^3} = 0$$

Thermodynamics and Statistics Group A.
Solutions:

A1. Isobarically



$$Q = nC_p\Delta T, \qquad C_p = \frac{5}{2}R$$

$$pV = nRT \quad \Rightarrow \quad \frac{pV}{R} = nT$$

$$\Rightarrow \; n\Delta T = \frac{p}{R}\Delta V$$

$$\Rightarrow \; Q = \frac{5}{2}R\left(\frac{p}{R}\Delta V\right) = \frac{5}{2}(100\ kPa)(1\text{x}10^{-3}\ m^3)$$

$$Q = 250\,J$$

A2. $dF = -SdT - pdV + \mu dN, \quad pV = nRT$

$$\Rightarrow F = -\int pdV\,, \qquad p = \frac{V_1}{V_1 + V_2}p_1$$

$$\Rightarrow F = NRT_1 \ln\frac{V_1 + V_2}{V_2}$$

A3. $C_{glass}m_{glass}(T_{glass} - T_f) = C_{water}m_{water}(T_f - T_{water})$

$$\Rightarrow \quad T_f = 297.20\ K = 24.2\ °C$$

A4. $Q = 1\ kg \text{ x } 4184\ {}^{J}/_{(kg\ K)} \text{ x } 20\ K + 1\ kg \text{ x } 334000\ {}^{J}/_{kg} = 417680\,J$

Thermodynamics and Statistics Group B.
Solutions:

B1. Since the entropy is a state function that only depends on the initial and final states, then the results are the same.

$$pV = nRT \quad \Rightarrow \quad \frac{V_f}{V_i} = \frac{T_f p_i}{T_i p_f} = \frac{(273+50)\text{x}100\ k}{(273+100)\text{x}200\ k}$$

$$\Delta S = C_p \ln\frac{T_f}{T_i} - R\ln\frac{p_f}{p_i} = R\ln\left(\frac{T_f}{T_i}\right)^{\frac{5}{2}}\left(\frac{p_i}{p_f}\right) \approx -0.7507R \approx -6.24\ {}^{J}/_{K}$$

B2. (a) $P(s=2) = \frac{1}{6}\times\frac{1}{6} = \frac{1}{36}$, $\quad P(s=3) = \frac{1}{6}\times\frac{1}{6}\times 2 = \frac{2}{36} = \frac{1}{18}$,
$P(s=4) = \frac{1}{6}\times\frac{1}{6}\times 3 = \frac{3}{36} = \frac{1}{12}$, $\quad P(s=5) = \frac{1}{6}\times\frac{1}{6}\times 4 = \frac{4}{36} = \frac{1}{9}$,
$P(s=6) = \frac{1}{6}\times\frac{1}{6}\times 5 = \frac{5}{36}$, $\quad P(s=7) = \frac{1}{6}\times\frac{1}{6}\times 6 = \frac{6}{36} = \frac{1}{6}$,
$P(s=8) = \frac{1}{6}\times\frac{1}{6}\times 5 = \frac{5}{36}$, $\quad P(s=9) = \frac{1}{6}\times\frac{1}{6}\times 4 = \frac{4}{36} = \frac{1}{9}$,
$P(s=10) = \frac{1}{6}\times\frac{1}{6}\times 3 = \frac{3}{36} = \frac{1}{12}$, $\quad P(s=11) = \frac{1}{6}\times\frac{1}{6}\times 2 = \frac{2}{36} = \frac{1}{18}$
$P(s=12) = \frac{1}{6}\times\frac{1}{6} = \frac{1}{36}$
(b) $s = 7$, $\qquad 7 = 1+6 = 2+5 = 3+4$
(c) $\langle s\rangle = 2\times\frac{1}{36} + 3\times\frac{2}{36} + 4\times\frac{3}{36} + 5\times\frac{4}{36} + 6\times\frac{5}{36} + 7\times\frac{6}{36} + 8\times\frac{5}{36} + 9\times\frac{4}{36} + 10\times\frac{3}{36} + 11\times\frac{2}{36} + 12\times\frac{1}{36} = \frac{252}{36} = 7$

$$\sigma = \sqrt{\sum_{s=2}^{12} P_s(s-\langle s\rangle)^2} = \sqrt{\frac{210}{36}} = \sqrt{\frac{35}{6}}$$

B3.
$Q = \Delta U - W = \Delta U + p|_{T=constant}\Delta V = \Delta U + \Delta(pV) = \Delta(U+pV) = \Delta H\ (Enthalpy)$

B4. Set volume V, temperature T, amount N, then for monatomic gas:
$C_V = \frac{3}{2}R$, $\ C_p = \frac{5}{2}R$ and $\ pV = NRT$
Adiabatic process: $Q = dU - \delta W = 0 \quad \Rightarrow \quad \Delta U + pdV = 0$

$$dU = NC_V dT, \quad Vdp + pdV = NRdT$$

$$\Rightarrow \quad \frac{C_V}{R}(Vdp + pdV) + pdV = 0$$

$$\Rightarrow \quad \frac{C_V}{R}Vdp = -\left(\frac{C_V}{R} + 1\right)pdV$$

$$\Rightarrow \quad \frac{dp}{p} = -\left(1 + \frac{R}{C_V}\right)\frac{dV}{V}$$

$$\Rightarrow \quad \ln\frac{p}{p_0} = \ln\left(\frac{V_0}{V}\right)^{\left(1+\frac{R}{C_V}\right)}$$

$$\Rightarrow \quad pV^{\left(1+\frac{R}{C_V}\right)} = pVV^{\frac{R}{C_V}} = NRTV^{\frac{R}{C_V}} = constant$$

$$\Rightarrow NTV^{\frac{2}{3}} = constant$$

Electricity and Magnetism Group A.

Solutions:

A1. $\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{z}$, σ is negative.

$$W = -qE(z_2 - z_1) = \frac{q|\sigma|}{2\varepsilon_0}(z_2 - z_1)$$

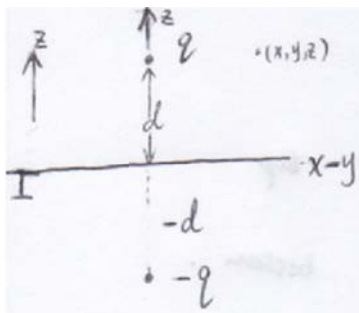
A2. $\vec{I} = aK_0\hat{y}$

A3. Volume charge density $\rho = \frac{Q}{V} = \frac{Q}{\frac{4}{3}\pi R^3} = \frac{3Q}{4\pi R^3}$

(a) $J = \rho v = \rho\omega r\sin\theta = \frac{3\omega Q r\sin\theta}{4\pi R^3}$

(b) $\theta = \frac{\pi}{2}$, $\Rightarrow$ $J = \frac{3\omega Q}{4\pi R^3}$

A4. (a)



$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\vec{x} + \vec{y} + (\vec{z} - d\hat{z})}{[x^2+y^2+(z-d)^2]^{3/2}} - \frac{q}{4\pi\varepsilon_0}\frac{\vec{x} + \vec{y} + (z+d)\hat{z}}{[x^2+y^2+(z+d)^2]^{3/2}}$$

(b) Unchanged

Electricity and Magnetism Group B.
Solutions:

B1. (a)

$$\vec{E}(z_0\hat{z}) = \frac{1}{4\pi\varepsilon_0 z_0^3}[3(p_1\hat{z}\cdot\hat{z})\hat{z} - p_1\hat{z}] = \frac{p_1}{2\pi\varepsilon_0 z_0^3}\hat{z}$$

(b)

$$\vec{F} = \nabla\vec{p}_2\cdot\vec{E}\Big|_{z=z_0} = -\frac{3p_1p_2\cos\theta}{2\pi\varepsilon_0 z_0^4}\hat{z}$$

(c)

$$\vec{\tau} = \vec{p}_2\times\vec{E} = \frac{p_1p_2\sin\theta}{2\pi\varepsilon_0 z_0^3}\hat{x}$$

(d) Along $\hat{z}$

B2. (a)

$$\rho_b = -\nabla\vec{P} = 0$$
$$\sigma_{b\,upper} = \vec{P}\cdot\hat{n} = P\hat{z}\cdot\hat{z} = P$$
$$\sigma_{b\,bottom} = \vec{P}\cdot\hat{n} = P\hat{z}\cdot(-\hat{z}) = -P$$

(b) Gauss's Law
Above: $\vec{E}_1 = 0$
Below: $\vec{E}_2 = 0$
Inside: $\vec{E}_3 = -\frac{P}{\varepsilon_0}\hat{z}$
(c) $E_{1t} - E_{2t} = 0$

Top: $E_{2n} - E_{1n} = \frac{P}{\varepsilon_0} = \frac{\sigma_{b\,upper}}{\varepsilon_0}$
Bottom: $E_{2n} - E_{1n} = -\frac{P}{\varepsilon_0} = \frac{\sigma_{b\,bottom}}{\varepsilon_0}$

B3. (a) Ampere's Law $\oint\vec{B}\cdot d\vec{l} = \mu_0 I$, cylindrical symmetry.
Inside $s < R$:

$$B2\pi s = \mu_0\int_0^s J_0\frac{r}{R}2\pi r dr$$
$$\vec{B}(\vec{s}) = \frac{\mu_0 J_0}{3R}s^2\hat{\varphi}$$

(b)

$$\vec{A} = \frac{\mu_0}{4\pi}\int\frac{\vec{J}d^3r'}{|\vec{r}-\vec{r}'|} \quad\Rightarrow\quad \vec{A} = A_z\hat{z}$$
$$\vec{B} = \nabla\times\vec{A} \quad\Rightarrow\quad -\frac{\partial A_z}{\partial s} = \frac{\mu_0 J_0}{3R}s^2$$
$$\Rightarrow\quad \vec{A}(s) = -\int_{s_0}^{s}\frac{\mu_0 J_0}{3R}r^2 dr = \frac{\mu_0 J_0}{9R}(s_0^3 - s^3)\hat{z}$$

B4. (a) Inside $s < a$, then Ampere's Law and Cylindrical symmetry.

$$\oint \vec{H} \cdot d\vec{l} = I_{free} \quad \Rightarrow \quad H2\pi s = \frac{I}{\pi a^2}\pi s^2 \quad \Rightarrow \quad \vec{H}(\vec{s}) = \frac{Is}{2\pi a^2}\hat{\varphi}$$

$$\vec{B} = \mu_0\left(1 + \chi_m\right)\vec{H} = \frac{\mu_0 I\left(H\chi_m\right)s}{2\pi a^2}\hat{\varphi}$$

(b) Outside $s > a$, then

$$\oint \vec{H} \cdot d\vec{l} = I_{free} \quad \Rightarrow \quad H2\pi s = I \quad \Rightarrow \quad \vec{H}(\vec{s}) = \frac{I}{2\pi s}\hat{\varphi}$$

$$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\varphi}$$

(c) For $s = a$

$$\vec{K} = \vec{M} \times \hat{s} = \chi_m \vec{H} \times \hat{s} = \frac{\chi_m I}{2\pi a}\hat{z} \quad \Rightarrow \quad B_{2n} - B_{1n} = \mu_0 K$$

Quantum Mechanics Group A.
Solutions:

A1. (a) $\langle\psi|\psi\rangle = 1 \quad ⇨ \quad C^2[9+16] = 1 \quad ⇨ \quad C = \frac{1}{5}$

(b) $P(m=-4) = |-4iC|^2 = \frac{16}{25}$

(c) $\vec{L}^2 = l(l+1)\hbar^2 = 4(4+1)\hbar^2 = 20\hbar^2, \quad P = 100\%$

A2.

$$\lambda = \frac{h}{p_e} = \frac{h}{p_n}, \quad p = \sqrt{2mE} = mv \quad ⇨ \quad v_n = \frac{\sqrt{2m_eE_e}}{m_n}$$

$$v_n = \frac{\sqrt{2\text{x}9.109\text{x}10^{-31}\ kg\ \text{x}5\text{x}10^3\text{x}1.602\text{x}10^{-19}\ J}}{1.673\text{x}10^{-27}\ kg} = 2.283\text{x}10^4\ {}^{m}/_{s}$$

A3. (a) $n = 0 \quad ⇨ \quad E_n = 0 \quad ⇨ \quad p = \sqrt{2mE_n} = 0 \quad ⇨ \quad \Delta x \geq \frac{\hbar}{2\Delta p} \to \infty$, out of the well.

(b) $\psi_n(x) = \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a} \quad ⇨ \quad \psi_o = 0 \quad ⇨ \quad P = |\psi_o|^2 = 0$, not existing.

A4.

$$E_{n_1n_2n_3} = \frac{\pi^2\hbar^2}{2ma^2}(n_1^2+n_2^2+n_3^2), \qquad n_i = 1,2,3,\cdots$$

$$E_1 = E_{111} = \frac{3\,\pi^2\hbar^2}{2ma^2}, \quad d_1 = 1$$

$$E_2 = E_{112} = E_{121} = E_{211} = \frac{3\,\pi^2\hbar^2}{ma^2}, \quad d_2 = 3$$

$$E_3 = E_{122} = E_{221} = E_{212} = \frac{9\,\pi^2\hbar^2}{2ma^2}, \quad d_3 = 3$$

Quantum Mechanics Group B.
Solutions:

B1.

$$[\hat{H},\hat{x}] = \left[\frac{\hat{p}^2}{2m} + V(x),\hat{x}\right] = \left[\frac{\hat{p}^2}{2m},\hat{x}\right] + [V(x),\hat{x}] = \left[\frac{\hat{p}^2}{2m},\hat{x}\right] = \frac{1}{2m}[\hat{p}^2,\hat{x}]$$

$$[\hat{H},\hat{x}] = \frac{1}{2m}\{[\hat{p},\hat{x}]\hat{p} + \hat{p}[\hat{p},\hat{x}]\} = -\frac{i\hbar}{m}\hat{p} = -\frac{\hbar^2}{m}\nabla$$

Thus, it is Hermitian.

B2. (a) $I = 2m\left(\frac{d}{2}\right)^2 = \frac{1}{2}md^2 = \frac{35}{2}m_p d^2 = \frac{35}{2}\text{x}1.673\text{x}10^{-17}\,kg \text{ x } (1.42\text{x}10^{-10}\,m)^2$

$$I = 5.904\text{x}10^{-46}\,kg\cdot m^2$$

(b) $B = \frac{\hbar^2}{2I} = \frac{(1.055\text{x}10^{-34}\,J\cdot s)^2}{2\text{x}5.904\text{x}10^{-46}\,kg\cdot m^2} = 9.426\text{x}10^{-24}\,J = 9.426\text{x}10^{-24}\,\frac{kg\,m^2}{s^2}$

(c) $H = \frac{L^2}{2I} \quad ⇨ \quad E_l = \frac{\hbar^2}{2I}l(l+1) \quad ⇨ \quad \Delta E = E_{l=8} - E_{l=2} = 66\,B$

$$⇨ E_{photon} = \frac{66\text{x}9.426\text{x}10^{-24}}{1.602\text{x}10^{-19}}eV = 38.834\,meV$$

(d) $E_{photon} = \hbar\omega = hv \quad ⇨ \quad v = \frac{E_{photon}}{h} = 9.389\text{x}10^{-11}\,Hz$

B3. (a) Even: $\psi_I(x) = Ce^{Kx}, \quad \psi_{II}(x) = B\cos(kx), \quad \psi_{III}(x) = Ce^{-Kx}$
Odd: $\psi_I(x) = De^{Kx}, \quad \psi_{II}(x) = A\sin(kx), \quad \psi_{III}(x) = -De^{-Kx}$

$$k = \sqrt{\frac{2mE}{\hbar^2}}, \quad K = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

(b) Boundary conditions:

$$\psi_{III}\left(\frac{a}{2}\right) = \psi_{II}\left(\frac{a}{2}\right) \quad and \quad \psi'_{III}\left(\frac{a}{2}\right) = \psi'_{II}\left(\frac{a}{2}\right)$$

Odd:

$$-De^{-K\frac{a}{2}} = A\sin\left(k\frac{a}{2}\right), \quad and \quad KDe^{-K\frac{a}{2}} = kA\cos\left(k\frac{a}{2}\right)$$

$$⇨ \; k\cot\left(k\frac{a}{2}\right) = -K$$

Even:

$$Ce^{-K\frac{a}{2}} = B\cos\left(k\frac{a}{2}\right), \quad and \quad -KCe^{-K\frac{a}{2}} = -kB\sin\left(k\frac{a}{2}\right)$$

$$⇨ \; k\tan\left(k\frac{a}{2}\right) = K$$

B4. (a) $\langle\psi|\psi\rangle = 1 \quad ⇨ \quad A^2(1+1) = 1 \quad ⇨ \quad A = \frac{1}{\sqrt{2}}$

(b) $\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}} = Bf(t) \quad ⇨ \quad f(t) = \frac{1}{B}\sqrt{\frac{\hbar}{2m\omega}}$

$\langle p\rangle = 0 \quad ⇨ \quad g(t) = 0$