UNL Department of Physics and Astronomy

Preliminary Examination - Day 1

May 9, 2013

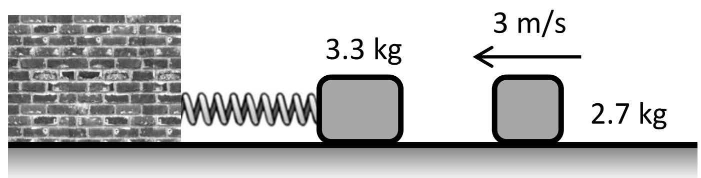
This test covers the topics of *Mechanics* (Topic 1) and *Thermodynamics and Statistical Mechanics* (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

Mechanics Group A - Answer only two Group A questions

A1 A mass of 2.7 kg sliding with a velocity of 3 m/s to the left collides inelastically with and sticks to a stationary mass of 3.3 kg attached to a spring. The spring constant is 450 N/m. Assume that sliding friction is negligible.

a. What is the amplitude of the resulting oscillation?

b. What is the period of this oscillation of the two masses as stuck together?



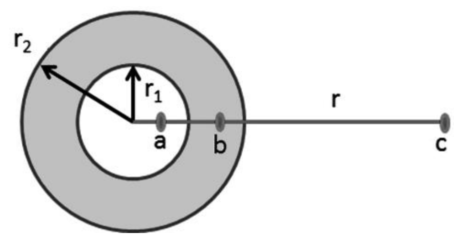
A2 Consider a spherical shell of finite thickness with inner radius r_1 and outer radius r_2 as shown in the diagram, having a uniform mass density ρ.

Calculate the gravitational acceleration at

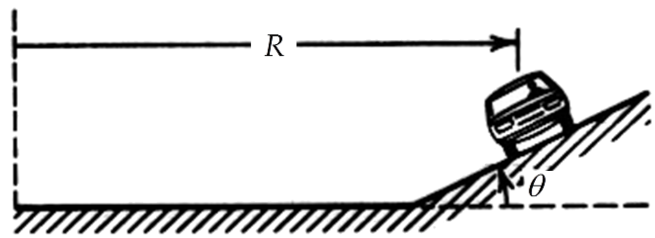
a. $r < r_1$

b. $r_1 < r < r_2$

c. $r > r_2$



A3 A car on a highway enters a curve of radius $R = 500$ m and banking angle $\theta = 30°$, as shown in the figure. The coefficient of friction between the wheels and the road is $\mu = 0.1$. What are the minimum and maximum speeds with which the car can round the curve without skidding sideways?



A4 Consider a force function $\mathbf{F}(x, y, z) = (ax + by^2)\mathbf{i} + cxy\,\mathbf{j}$.

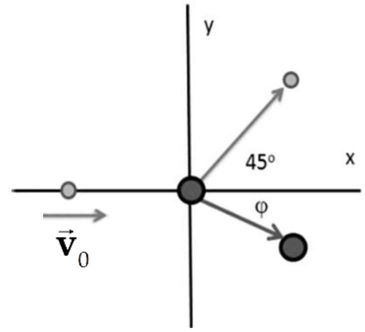
a. For what values of the constants a, b, and c is the force conservative?

b. Calculate the path integral $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ from point A(0,0,0) to point B(1,1,1) using the path $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$ (for $0 \le t \le 1$).

c. Calculate the path integral $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ from point A(0,0,0) to point B(1,1,1) using the path $\mathbf{r} = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$ (for $0 \le t \le 1$).

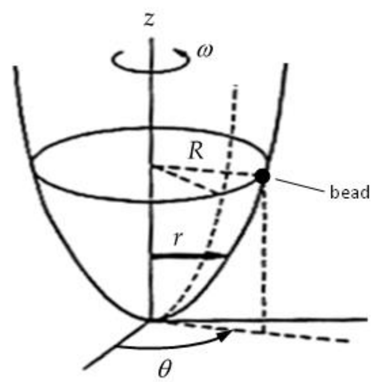
Mechanics Group B - Answer only two Group B questions

B1 A proton (mass m_p) with initial velocity $\vec{\mathbf{v}}_0$ collides with a helium atom (mass $4m_p$) that is initially at rest. If the proton leaves the point of impact at an angle of 45° with its original line of motion, find the final velocities of each particle. Assume that the collision is perfectly elastic.



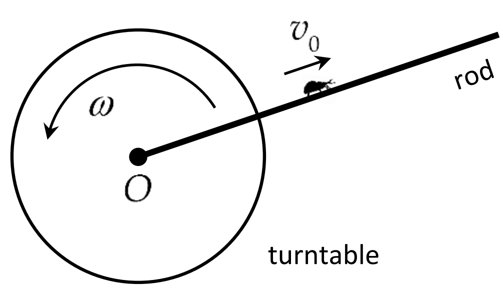
B2 A bead slides along a smooth wire bent in the shape of a parabola $z = cr^2$ (see figure). The bead rotates in a circle of radius R when the wire is rotating about its vertical symmetry axis with angular velocity ω.

a. Find the Lagrangian as being dependent only on r.
b. Find the equation of motion.
c. Find the value of c.



$U = 0$ at $z = 0$

B3 A rod is fixed on a turntable in a radial direction starting at O. A bug starts crawling on the rod (radially outward from O) with constant speed v_0. The bug can hold on to the rod with a maximum force equal to its weight on Earth. The turntable rotates with an angular frequency ω. How far from O can the bug crawl before it loses its grip?



B4 A particle of mass m moves in two dimensions under the following potential energy function:

$$V(\vec{\mathbf{r}}) = \tfrac{1}{2}k(x^2 + 4y^2).$$

Find the resulting trajectory, given the initial conditions at $t = 0$: $x = a$, $y = 0$, $\dot{x} = 0$, $\dot{y} = v_0$.

Thermodynamics and Statistical Mechanics Group A - Answer only two Group A questions

A1 Suppose you have a bag of 100 marbles, of which 50 are red and 50 are blue. You are allowed to draw four marbles from each bag according to the following rules:

1. draw one marble, record its color, and return it to the bag;
2. shake the bag and then draw another marble;
3. continue this process until you have drawn and returned four marbles.

What is the most likely color state (N,M) of the drawn four-marble system where N stands for the number of the red marbles and M for the number of the blue marbles? What is the probability of this state to be drawn?

A2 A weightlifter snatches a barbell with mass $m = 180.0$ kg and moves it a distance $h = 1.25$ m vertically upward. If we consider the weightlifter to be a thermodynamical system, how much heat must he give off if his internal energy decreases by 4000 joules?

A3 One mole of an ideal monatomic gas expands isothermally, at 300 K, so that its volume increases from 10 to 20 liters. Calculate

a. the work done by the gas on its surroundings;
b. the amount of heat absorbed by it; and
c. the entropy change of the gas in this process.

A4 At low temperatures, the heat capacity at constant pressure for a piece of solid is a cubic function of temperature: $C_p = AT^3$, where A is a constant. Derive the expressions for the entropy S and Gibbs free energy G of this piece of solid in this temperature range.

Thermodynamics and Statistical Mechanics Group B - Answer only two Group B questions

B1 Consider a Carnot cycle performed on an ideal gas. In each cycle, the gas absorbs 3000 J of heat from a thermal reservoir at $T_h = 500\ \mathrm{K}$ and discards heat to a thermal reservoir at $T_c = 325\ \mathrm{K}$.

a. Find the total work done by the gas in one cycle by computing the work done by it in each of the two adiabatic and two isothermal processes comprising the cycle.

b. Subsequently, compute the efficiency of the heat engine corresponding to this particular cycle with the given ideal gas by dividing the work done over the full cycle divided by the total heat absorbed.

B2 Find an entropy change of an ideal monoatomic gas of N particles occupying a volume V_1 when it expands to a volume V_2 under constant pressure.

B3 A spin-1/2 paramagnet can be modeled as a collection of N independent two-level systems with a separation $\Delta = 2\mu H$ between the levels, where μ is the magnetic moment and H the applied magnetic field. Calculate the magnetic susceptibility of this system at temperature T.

B4 The equation of state for a non-ideal gas can be approximated as $P = \frac{NkT}{V}\left(1 + \frac{B_2(T)}{V}\right)$, where the so-called "second virial coefficient" B_2 is some function of temperature.

a. Find the expression for $\left(\frac{\partial S}{\partial V}\right)_T$.

b. Compute $\left(\frac{\partial U}{\partial V}\right)_T$. *Hint: you may find the expression derived in the previous step useful.*

EQUATIONS THAT MAY BE HELPFUL

Rotating reference frame: $\vec{\mathbf{a}}' = \vec{\mathbf{a}} - 2\vec{\omega}\times\vec{\mathbf{v}}' - \dot{\vec{\omega}}\times\vec{\mathbf{r}}' - \vec{\omega}\times(\vec{\omega}\times\vec{\mathbf{r}}')$

General efficiency η of a heat engine producing work $|W|$ while taking in heat $|Q_\text{h}|$ is $\eta = \frac{|W|}{|Q_\text{h}|}$.

For a Carnot cycle operating as a heat engine between reservoirs at T_h and at T_c the efficiency becomes $\eta_\text{C} = \frac{T_\text{h} - T_\text{c}}{T_\text{h}}$.

Clausius' theorem: $\sum_{i=1}^{N} \frac{Q_i}{T_i} \le 0$, which becomes $\sum_{i=1}^{N} \frac{Q_i}{T_i} = 0$ for a reversible cyclic process of N steps.

For adiabatic processes in an ideal gas with constant heat capacity, $PV^\gamma = \text{constant}$.

$$\frac{dP}{dT} = \frac{\lambda}{T\Delta V}$$

specific heat of water: 4186 J/(kg·K)
latent heat of ice melting: 334 J/g

$H = E + PV \qquad F = E - TS \qquad G = F + PV \qquad \Omega = F - \mu N$

$dE = TdS - PdV + \mu dN \qquad dS = dE/T + PdV/T - \mu dN/T \qquad dH = TdS + VdP + \mu dN$

$dF = -SdT - PdV + \mu dN \qquad dG = -SdT + VdP + \mu dN \qquad d\Omega = -SdT - PdV - Nd\mu$

E.1 ALGEBRAIC FUNCTIONS

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right), \qquad \left|\tan^{-1}\left(\frac{x}{a}\right)\right| < \frac{\pi}{2} \qquad \text{(E.1)}$$

$$\int \frac{x\,dx}{a^2 + x^2} = \frac{1}{2}\ln(a^2 + x^2) \qquad \text{(E.2)}$$

$$\int \frac{dx}{x(a^2 + x^2)} = \frac{1}{2a^2}\ln\left(\frac{x^2}{a^2 + x^2}\right) \qquad \text{(E.3)}$$

$$\int \frac{dx}{a^2x^2 - b^2} = \frac{1}{2ab}\ln\left(\frac{ax - b}{ax + b}\right) \qquad \text{(E.4a)}$$

$$= -\frac{1}{ab}\coth^{-1}\left(\frac{ax}{b}\right) \qquad \text{(E.4b)}$$

$$= -\frac{1}{ab}\tanh^{-1}\left(\frac{ax}{b}\right), \qquad a^2x^2 < b^2 \qquad \text{(E.4c)}$$