UNL - Department of Physics and Astronomy

# Preliminary Examination - Day 1
# May 14, 2015

This test covers the topics of *Quantum Mechanics* (Topic 1) and *Electrodynamics* (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

**WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY**

**Quantum Mechanics Group A** - Answer only two Group A questions

**A1** Show that $[xp, px] = 0$, where $x$ and $p$ are the position and momentum operators in one dimension, respectively.

**A2**

**Part a.** The operator $\mathsf{A}$ is defined by $\mathsf{A}\left[f(x)\right] = \left|f(x)\right|$. Is operator $\mathsf{A}$ linear? Explain.

**Part b.** The operator $\mathsf{B}$ is defined by $\mathsf{B}\left[f(x)\right] = \left(f(x)\right)^*$. Is operator $\mathsf{B}$ self-adjoint (Hermitian)? Explain.

**A3** A krypton fluoride (KrF) laser beam (wavelength $\lambda = 248$ nm) hits a polished metal surface. As a result, electrons leave the metal, the fastest ones having a speed of $7.96 \times 10^5$ m/s.

*a.* Find the energy per photon in the KrF laser beam, in eV.
*b.* Find the work function $\Phi$ of the metal, in eV.

**A4** The wavefunction of a free particle moving in one dimension is given by $\Psi(x) = A\sin(kx)$, where $A$ and $k$ are constants.

*a.* Is this wavefunction an energy eigenstate?
*b.* Is this wavefunction a momentum eigenstate?
*c.* The momentum of the particle is measured. What are possible outcomes of this measurement and what are the corresponding probabilities?

**Quantum Mechanics Group B** - Answer only two Group B questions

**B1** A particle has spin $\frac{1}{2}$ .We define the operators $S_\pm$ as $S_\pm = S_x \pm iS_y$. These two operators, and the operator $S_z$ are given by

$$S_+ = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad S_- = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad , \quad S_z = \tfrac{1}{2}\hbar\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

*a.* Is the operator $S_+S_-$ self-adjoint (Hermitian)? Explain.

*b.* Find the matrix representations of the operators $S_x$ and $S_y$ .

*c.* Find the normalized eigenvector of the operator $S_y$ for its eigenvalue $(+\frac{1}{2})\hbar$ .

*d.* Assume the particle is in this $S_y = (+\frac{1}{2})\hbar$ spin eigenstate. Then, the *z* component of the spin is measured, and is found to be $(-\frac{1}{2})\hbar$ . What was the probability to find this value?

*e.* Immediately after the measurement in part *d*., the *z* component of the spin is measured again. What is the probability that the value $(+\frac{1}{2})\hbar$ is now found?

**B2** A particle with mass $m$ in the harmonic oscillator potential (angular frequency $\omega$) has the initial state

$$\psi(x,0) = A\left[2\varphi_0(x) - \tfrac{3}{2}\sqrt{2}\,\varphi_1(x) + \sqrt{2}\,\varphi_2(x)\right],$$

where $A$ is a normalization constant. Here, the $\varphi_n(x)$ are normalized stationary states: $\hat{H}\varphi_n(x) = (n+\frac{1}{2})\hbar\omega\varphi_n(x)$ .

*a.* Calculate the expectation value of the energy.

*b.* Calculate the expectation value of the parity operator $\hat{P}$, which is defined by $\hat{P}\left[f(x)\right] = f(-x)$ .

At a later time $T$ , the wavefunction is

$$\psi(x,T) = B\left[2\varphi_0(x) - \tfrac{3}{2}i\sqrt{2}\,\varphi_1(x) - \sqrt{2}\,\varphi_2(x)\right]$$

for some constant $B$.

*c.* Calculate the smallest possible value of $T$.

**B3** A particle with mass $m$ moves in the one-dimensional delta-function potential $V(x) = -a\delta(x)$, with $a > 0$. The energy of the ground state is $E = -|E|$. We define the positive quantity $\kappa$ as $\kappa^2 \equiv \frac{2m|E|}{\hbar^2}$. Because of the delta-function potential, the derivative $\psi'(x)$ of the particle's wavefunction is not continuous at the origin. The kink is given by:

$$\lim_{\varepsilon \to 0}\left[\psi'(x=+\varepsilon) - \psi'(x=-\varepsilon)\right] = -\frac{2ma}{\hbar^2}\psi(x=0).$$

*a.* Calculate the ground state energy.
*b.* Calculate the normalized ground-state wavefunction.
*c.* Use a symmetry argument to prove that $\langle x \rangle = 0$.
*d.* Calculate $\langle x^2 \rangle$.
*e.* Calculate $\Delta x$, the uncertainty in $x$.
*f.* Use a symmetry argument to prove that $\langle p \rangle = 0$.
*g.* Calculate $\langle p^2 \rangle$.
*h.* Calculate $\Delta p$, the uncertainty in $p$.
*i.* Verify that the uncertainty principle holds for this case.

**B4** A particle is in the eigenstate of $\mathbf{L}^2$ and $L_z$ with the corresponding quantum numbers $\ell$ and $m$.

*a.* Using the commutation relations for the operators $L_x$, $L_y$, and $L_z$, prove that $\langle L_x \rangle = \langle L_y \rangle = 0$.

Measurement is made of the component of $\mathbf{L}$ along the $z'$ axis which makes an angle $\alpha$ with the $z$ axis.

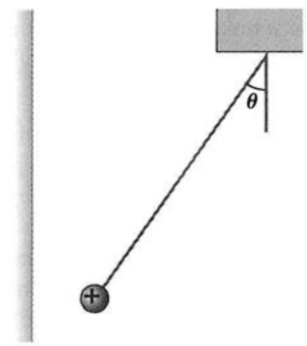
*b.* What is the expectation value of this component?
*c.* What is the expectation value of the square of this component?

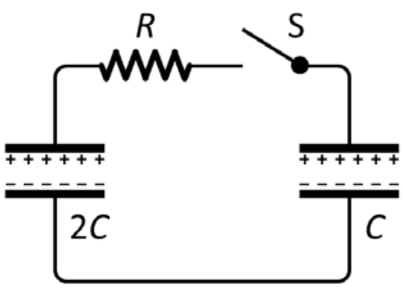
In part *c.*, use the equation $\langle L_x L_y + L_y L_x \rangle = 0$ (you don't have to prove this equation).

<u>**Electrodynamics Group A**</u> - <u>Answer only two Group A questions</u>

**A1** A small sphere with a mass of 0.002 g and carrying a charge of $5.00 \times 10^{-8}$ C hangs from a thread near a very large, charged insultating sheet, as shown in the Figure. The charge density on the sheet is $-2.50 \times 10^{-9}$ C/m$^2$.

Find the angle $\theta$ of the thread.




**A2** Two capacitors are wired together with a resistor (resistance $R$), and a switch S which is initially open, as shown. Initially, a charge $Q$ is deposited on *each* capacitor. The capacitance of the left capacitor is $2C$ and the capacitance of the right capacitor is $C$. Now the switch S is closed. When the system reaches equilibrium (a long time later), what is the charge stored on the left capacitor?

**A3** A square loop of wire with side length 10 cm is immersed in a uniform 0.6 T magnetic field. Initially, the magnetic field $\mathbf{B}$ and the loop's normal vector $\mathbf{A}$ point in the same direction.

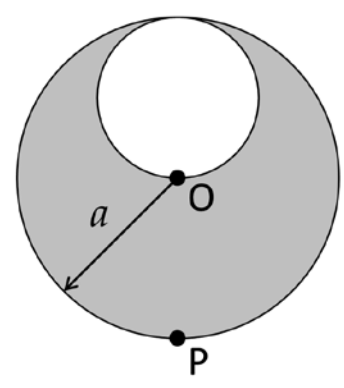
*a.* What is the magnetic flux through the loop?

The loop is now quickly flipped over, so that now $\mathbf{B}$ and $\mathbf{A}$ point in opposite directions.

*b.* If the resistance of the wire making up the loop is 15 Ω, what is the magnitude of the total charge that flows past a fixed point on the wire as the loop is flipped? *Hint: you may use a variant of Faraday's Law:* $\mathcal{E}_{\text{average}} = -\Delta(\Phi_B)/\Delta t$, *where the deltas indicate a change in the relevant quantity as opposed to an infinitesimal differential quantity.*

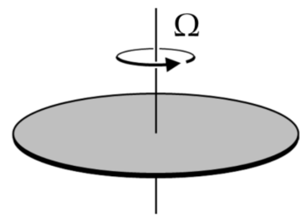
**A4** A long cylindrical conductor of radius $a$ has an off-center cylindrical hole of radius $a$/2 down its full length, as shown in the Figure. A current $I$ flows through the conductor into the page, with uniform current density.

What is the magnitude of the magnetic field at point P?



**Electrodynamics Group B** - Answer only two Group B questions

**B1** Charge $Q$ is uniformly distributed over a thin disc of radius $R$. The disc rotates with angular velocity $\Omega$.




*a.* Find the disc's magnetic moment *M*.

*b.* Find the magnetic field on the symmetry axis at the distance *x* from the disc's center.

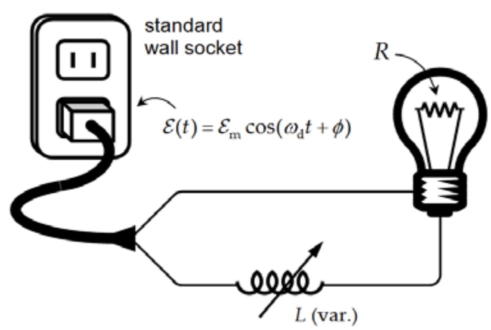
*c.* Prove that at $x \gg R$ the magnetic field behaves as $B = \mu_0 \dfrac{M}{2\pi x^3}$.

*Hint: use the binomial expansion* $(1+z)^{-1/2} = 1 - z/2 + 3z^2/8 + O(z^3)$.

**B2** A wooden (nonmagnetizable) ring with square cross-sectional area has an inner diameter of 6 cm, an outer diameter of 7 cm, and a height of ½ cm. Wire is wrapped tightly around the square cross section so that there are 100 turns per radian of the azimuthal coordinate of the toroidal winding. The wire carries a current of 1.5 A. What is the magnetic flux through the windings?


**B3** A thin-walled cylinder of length $L$ and radius $R$ has a charge $Q$ uniformly distributed (as an areal charge density) on its surface. What is the magnitude of the electric field on the cylinder's axis of symmetry at one of its ends (point P)?

**B4** A simple way to dim an incandescent light bulb is to connect it in series to an inductor with variable inductance as shown in the diagram. A plug connects this circuit to a standard U.S. wall socket, with $\mathcal{E}_\mathrm{m} = 170$ V and $\omega_\mathrm{d} = 2\pi \times 60$ rad/s. The light bulb is rated at 100 W. The amplitudes of the voltages through *R* and *L* are called $V_R$ and $V_L$, respectively.




*a.* Give expressions for $V_R$ and $V_L$ in terms of $I$, $\omega_\mathrm{d}$, $R$, and $L$, where *I* is the amplitude of the current in the circuit.

*b.* Find the impedance of the circuit in terms of $\omega_\mathrm{d}$, $R$, and $L$.

*c.* To make the light bulb burn at its maximum power (100 W), should the inductance *L* be made very large or very small? Why?

*d.* At what inductance should the inductor be set to dim the light bulb to 30 W?

## Physical Constants

| | |
|---|---|
| speed of light .................. $c = 2.998\times10^{8}$ m/s | electrostatic constant ... $k = (4\pi\varepsilon_0)^{-1} = 8.988\times10^{9}$ m/F |
| Planck's constant ........... $h = 6.626\times10^{-34}$ J·s | electron mass ............... $m_{\text{el}} = 9.109\times10^{-31}$ kg |
| Planck's constant / $2\pi$.... $\hbar = 1.055\times10^{-34}$ J·s | electron rest energy...... 511.0 keV |
| Boltzmann constant ...... $k_{\text{B}} = 1.381\times10^{-23}$ J/K | Compton wavelength .. $h/m_{\text{el}}c = 2.426$ pm |
| elementary charge ......... $e = 1.602\times10^{-19}$ C | proton mass ................ $m_{\text{p}} = 1.673\times10^{-27}\,\text{kg} = 1836 m_{\text{el}}$ |
| electric permittivity ...... $\varepsilon_0 = 8.854\times10^{-12}$ F/m | 1 bohr ............................ $a_0 = \hbar^2/ke^2 m_{\text{el}} = 0.5292$ Å |
| magnetic permeability ... $\mu_0 = 1.257\times10^{-6}$ H/m | 1 hartree (= 2 rydberg) ... $E_{\text{h}} = \hbar^2/m_{\text{el}}{a_0}^2 = 27.21$ eV |
| molar gas constant.......... $R = 8.314$ J / mol·K | gravitational constant ... $G = 6.674\times10^{-11}\ \text{m}^3/\text{kg s}^2$ |
| Avogrado constant ....... $N_{\text{A}} = 6.022\times10^{23}\ \text{mol}^{-1}$ | $hc$ ................................... $hc = 1240$ eV·nm |

## Equations That May Be Helpful

### QUANTUM MECHANICS

Energy levels in a one-dimensional, infinitely deep box of width $a$ :

$$E_n = n^2 \frac{\pi^2\hbar^2}{2ma^2}$$

Angular momentum:

$$[L_x, L_y] = i\hbar L_z \quad \text{et cycl.}$$

### ELECTROSTATICS

$$\oint\!\!\!\oint_S \mathbf{E}\cdot d\mathbf{A} = \frac{q_{\text{enc}}}{\varepsilon_0} \qquad \mathbf{E} = -\nabla V \qquad \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{E}\cdot d\ell = V(\mathbf{r}_1) - V(\mathbf{r}_2) \qquad V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Work done $W = -\int_{\mathbf{a}}^{\mathbf{b}} q\mathbf{E}\cdot d\ell = q\left[V(\mathbf{b}) - V(\mathbf{a})\right]$ Energy stored in elec. field: $W = \frac{1}{2}\varepsilon_0 \int_V E^2 d\tau = Q^2/2C$

Multipole expansion:

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r}\int \rho(\mathbf{r}')d\tau' + \frac{1}{r^2}\int r'\cos(\theta')\rho(\mathbf{r}')d\tau' + \frac{1}{r^3}\int (r')^2\left\{\tfrac{3}{2}\cos(\theta') - \tfrac{1}{2}\right\}\rho(\mathbf{r}')d\tau' + \ldots\right]$$

where the first term is the monopole term, the second is the dipole term, the third is the quadrupole term … ; $\mathbf{r}$ and $\mathbf{r}'$ are field point and source point and $\theta'$ is the angle between them.

$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$ $\quad \nabla\cdot\mathbf{D} = \rho_\text{f}$ $\quad \rho_\text{b} = -\nabla\cdot\mathbf{P}$ $\quad \sigma_\text{b} = \mathbf{P}\cdot\hat{\mathbf{n}}$

The above are true for *all* dielectrics. Confining ourselves to linear, isotropic, and homogeneous (LIH) dielectrics, we also have:

$\mathbf{D} = \varepsilon\mathbf{E}$ $\quad \mathbf{P} = \chi_\text{e}\varepsilon_0\mathbf{E}$ $\quad \varepsilon = \varepsilon_0(1+\chi_\text{e})$ $\quad \kappa_\text{e} = \varepsilon/\varepsilon_0$ $\quad \chi_\text{e} = \kappa_\text{e} - 1$

$C(\text{dielectric}) = \kappa_\text{e} C(\text{vacuum})$

Boundary condition: $\mathbf{E}_\text{above} - \mathbf{E}_\text{below} = \dfrac{\sigma}{\varepsilon_0}\hat{\mathbf{n}}$

**MAGNETOSTATICS**

Lorentz Force: $\mathbf{F} = q\mathbf{E} + q(\mathbf{v}\times\mathbf{B})$ Current densities: $I = \int \mathbf{J}\cdot d\mathbf{A}$, $I = \int \mathbf{K}\cdot d\ell$

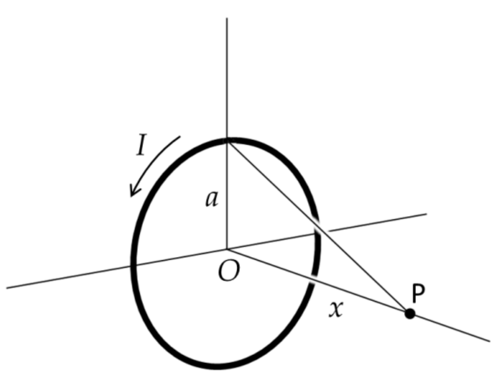
Biot-Savart Law: $\mathbf{B}(\mathbf{r}) = \dfrac{\mu_0}{4\pi}\displaystyle\int \frac{I d\ell\times\hat{\mathbf{R}}}{R^2}$ ($\mathbf{R}$ is vector from source point to field point $\mathbf{r}$)

For surface currents: $\mathbf{B}(\mathbf{r}) = \dfrac{\mu_0}{4\pi}\displaystyle\int \frac{\mathbf{K}\times\hat{\mathbf{R}}}{R^2}da$

For straight wire segment: $B = \dfrac{\mu_0 I}{4\pi s}\left[\sin\theta_2 - \sin\theta_1\right]$ where $s$ is the perpendicular distance from wire.

Infinitely long solenoid: $B$-field inside is $B = \mu_0 n I$ ($n$ is number of turns per unit length)

Ampere's law: $\oint \mathbf{B}\cdot d\ell = \mu_0 I_\text{enclosed}$




Field of loop at P, on axis, at distance $x$ from center:

$$\left|\mathbf{B}_\text{P}\right| = \mu_0\frac{Ia^2}{2(x^2+a^2)^{3/2}}$$

## Magnetic vector potential A

$\mathbf{B} = \nabla \times \mathbf{A}$

$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \quad \Rightarrow \quad \mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}}{r - r'} d\tau'$

For line and surface currents $\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{I}{r - r'} d\ell \qquad \mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}}{r - r'} da'$

From Stokes' theorem $\oint \mathbf{A} \cdot d\ell = \int \mathbf{B} \cdot d\mathbf{a}$

For a magnetic dipole $\mathbf{m}$, $\mathbf{A}_{\text{dipole}} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}$

## Magnetic dipoles

Magnetic dipole moment of a current distribution is given by $\mathbf{m} = I \int d\mathbf{a}$ .

Force on magnetic dipole: $\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$

Torque on magnetic dipole: $\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}$

*B*-field of magnetic dipole: $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{1}{r^3} \left[ 3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m} \right]$

The dipole-dipole interaction energy is $U_{\text{DD}} = \frac{\mu_0}{4\pi R^3} \left[ (\mathbf{m}_1 \cdot \mathbf{m}_2) - 3(\mathbf{m}_1 \cdot \hat{\mathbf{R}})(\mathbf{m}_2 \cdot \hat{\mathbf{R}}) \right]$, where $\mathbf{R} = \mathbf{r}_1 - \mathbf{r}_2$.

## Material with magnetization M

produces a magnetic field equivalent to that of (bound) volume and surface current densities

$\mathbf{J}_{\text{b}} = \nabla \times \mathbf{M}$ and $\mathbf{K}_{\text{b}} = \mathbf{M} \times \hat{\mathbf{n}}$

$\oint \mathbf{H} \cdot d\ell = I_{\text{free, enclosed}} \qquad \mathbf{H} = \mathbf{B} / \mu_0 - \mathbf{M}$

For linear magnetic material $\mathbf{M} = \chi_{\text{m}} \mathbf{H}$ and $\mathbf{B} = \mu_0 (1 + \chi_{\text{m}}) \mathbf{H}$ or $\mathbf{B} = \mu \mathbf{H}$

Boundary conditions: $\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0 (\mathbf{K} \times \hat{\mathbf{n}})$

**Maxwell's Equations in vacuum**

1. $\nabla \cdot \mathbf{E} = \dfrac{\rho}{\varepsilon_0}$ Gauss' Law
2. $\nabla \cdot \mathbf{B} = 0$
3. $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ Faraday's Law
4. $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \varepsilon_0 \mu_0 \dfrac{\partial \mathbf{E}}{\partial t}$ Ampere's Law with Maxwell's correction

**Maxwell's Equations in linear, isotropic, and homogeneous (LIH) media**

1. $\nabla \cdot \mathbf{E} = \dfrac{\rho_f}{\varepsilon}$ Gauss' Law
2. $\nabla \cdot \mathbf{B} = 0$
3. $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ Faraday's Law
4. $\nabla \times \mathbf{B} = \mu \mathbf{J}_f + \varepsilon \mu \dfrac{\partial \mathbf{E}}{\partial t}$ Ampere's Law with Maxwell's correction

**Induction**

Alternative way of writing Faraday's Law: $\oint \mathbf{E} \cdot d\ell = -\dfrac{d\Phi}{dt}$

Mutual and self inductance: $\Phi_2 = M_{21} I_1$, and $M_{21} = M_{12}$; $\Phi = LI$

Energy stored in magnetic field: $W = \frac{1}{2} \mu_0^{-1} \int_V B^2 d\tau = \frac{1}{2} LI^2 = \frac{1}{2} \oint \mathbf{A} \cdot \mathbf{I} \, d\ell$

## VECTOR DERIVATIVES

**Cartesian.** $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

*Gradient :* $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

*Divergence :* $$\nabla\cdot\mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

*Curl :* $$\nabla\times\mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

*Laplacian :* $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

**Spherical.** $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

*Gradient :* $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

*Divergence :* $$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

*Curl :* $$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

*Laplacian :* $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$$

**Cylindrical.** $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

*Gradient :* $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

*Divergence :* $$\nabla\cdot\mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$$

*Curl :* $$\nabla\times\mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$$

*Laplacian :* $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$$

## VECTOR IDENTITIES

**Triple Products**

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

**Product Rules**

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

**Second Derivatives**

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

## FUNDAMENTAL THEOREMS

**Gradient Theorem :** $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

**Divergence Theorem :** $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

**Curl Theorem :** $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

**INTEGRALS**

| $f(x)$ | | $\int_0^\infty f(x)\,dx$ |
|---|---|---|
| $e^{-a x^2}$ | .............. | $\frac{\sqrt{\pi}}{2\sqrt{a}}$ |
| $x e^{-a x^2}$ | .............. | $\frac{1}{2a}$ |
| $x^2 e^{-a x^2}$ | .............. | $\frac{\sqrt{\pi}}{4 a^{3/2}}$ |
| $x^3 e^{-a x^2}$ | .............. | $\frac{1}{2a^2}$ |
| $x^4 e^{-a x^2}$ | .............. | $\frac{3\sqrt{\pi}}{8 a^{5/2}}$ |
| $x^5 e^{-a x^2}$ | .............. | $\frac{1}{a^3}$ |
| $x^6 e^{-a x^2}$ | .............. | $\frac{15\sqrt{\pi}}{16 a^{7/2}}$ |

$$\int_0^\infty \frac{1}{1+ay^2}\,dy = \pi / 2a^{1/2}$$

$$\int_0^\infty y^n e^{-ay}\,dy = \frac{n!}{a^{n+1}}$$

$$\int \frac{r^3 dr}{(x^2+r^2)^{3/2}} = (r^2+x^2)^{1/2} + \frac{x^2}{(r^2+x^2)^{1/2}}$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) \qquad \left|\tan^{-1}\left(\frac{x}{a}\right)\right| < \frac{\pi}{2}$$

$$\int \frac{xdx}{a^2+x^2} = \frac{1}{2}\ln\left(a^2+x^2\right)$$

$$\int \frac{dx}{x(a^2+x^2)} = \frac{1}{2a^2}\ln\left(\frac{x^2}{a^2+x^2}\right)$$

$$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right)$$

$$= -\frac{1}{ab}\coth^{-1}\left(\frac{ax}{b}\right)$$

$$= -\frac{1}{ab}\tanh^{-1}\left(\frac{ax}{b}\right), \quad a^2x^2 < b^2$$