UNL - Department of Physics and Astronomy

Preliminary Examination - Day 2
May 15, 2015

This test covers the topics of *Thermodynamics and Statistical Mechanics* (Topic 1) and *Mechanics* (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY

Thermodynamics and Statistical Mechanics Group A - Answer only two Group A questions

A1 You decide to tell your fortune by drawing two cards from a standard full deck of 52 cards. What is the probability of drawing two cards of the same suit in a row (for instance, two hearts)? Explain your answer.

A2 Consider a diatomic gas near room temperature. What fraction of the supplied heat is converted into work done by the gas in its expansion at constant pressure? Ignore vibrational degrees of freedom of the gas molecules.

A3 One mole of an ideal gas initially fills one of the two equal chambers of a thermally insulated vessel. The second chamber is evacuated. After a partition between the chambers is punctured, the gas fills the entire vessel. Find the entropy change of the gas in this process.

A4 A reversible Carnot cycle is used to maintain the contents of a refrigerator at 4°C, while the room temperature is 20°C. Heat flow through the walls of the refrigerator amounts to 30 W. How much energy (in kilowatt-hours) must be supplied from the power source per day?

Thermodynamics and Statistical Mechanics Group B - Answer only two Group B questions

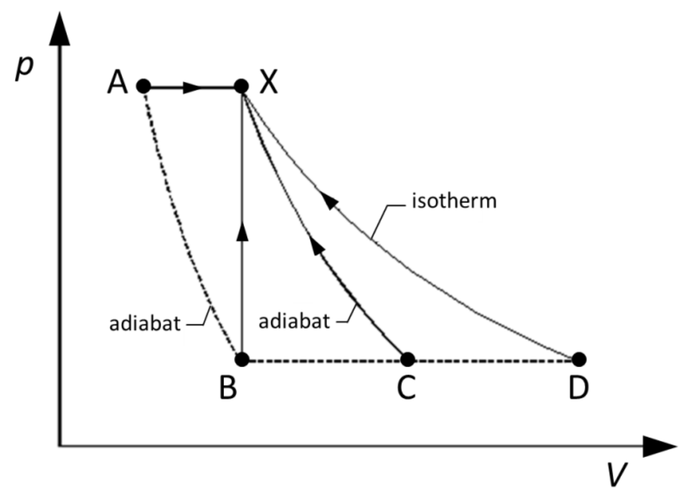
B1

Part *a.* - A quantity of water, initially at 10°C, is brought into contact with a heat reservoir at a temperature of 90°C. What is the entropy change of the entire system when the water reaches the temperature of the bath? Express the answer in terms of the heat capacity, C, of the water, and assume that C doesn't depend on temperature.

Part *b.* - What is the entropy change of the entire system if we use two reservoirs to heat the water first from 10°C to 50°C, and then from 50°C to 90°C?

Part *c.* - Is it possible, in principle, to heat the water from an initial temperature T_i to the final temperature T_f with no change at all in the entropy of the entire system?

B2 For a certain amount of ideal gas, consider four reversible processes represented by the following lines in the Figure: AX, BX, CX, and DX, all ending at point X. (The dashed lines show the relative positions of the points.) Rank the amounts of heat received by the gas in these processes. Explain your reasoning.



B3 Two walkers move in steps on a two-dimensional square lattice, starting at the same vertex. At each step, a walker moves from one vertex to an adjacent one, choosing one of the four directions at random. The length of the side of each cell of the lattice is 1. What is the mean-squared distance between the two walkers after each makes N steps?

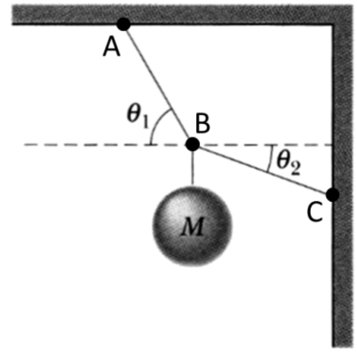
B4 Supercooled water at –2°C in a thermally insulated container suddenly transforms into an equilibrium ice-water mixture.

a. What fraction of water has turned into ice?

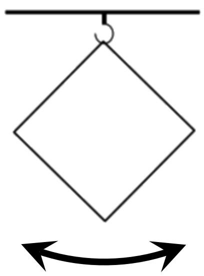
b. How much was the entropy change per gram?

Mechanics Group A - Answer only two Group A questions

A1 The system in the Figure is in equilibrium. The angles are $\theta_1 = 60°$ and $\theta_2 = 20°$, and the ball has mass $M = 2.0$ kg.

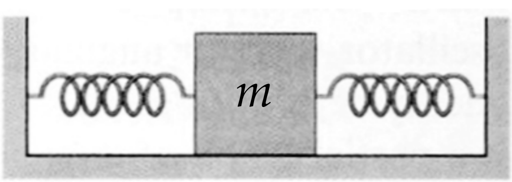


a. What is the tension in string AB?
b. What is the tension in string BC?



A2 Thin, rigid, uniform steel wire of linear mass density 20 g/cm is used to form a square of length 10 cm on a side. It is suspended as shown and executes pendulum-like motion (in the plane of the square) with a small amplitude. What is the frequency of this motion?

A3 Suppose you could drill a hole through the Earth and then drop into it. How long would it take you to pop up on the other side of the Earth?

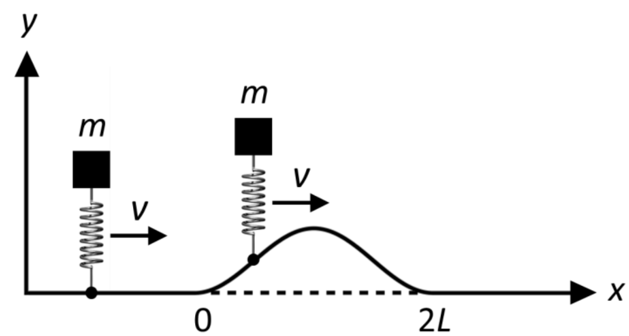


A4 In the Figure, two springs are attached to a block that can oscillate over a frictionless floor. If the left spring is removed, the block oscillates at a frequency of 30 Hz. If, instead, the spring on the right is removed, the block oscillates at a frequency of 45 Hz.

At what frequency does the block oscillate with both springs attached?

Mechanics Group B - Answer only two Group B questions

B1 A car is traveling in the x-direction and maintains a constant horizontal speed v. The car goes over a bump whose shape is described by $y_0 = A\left(1-\cos(\pi x/L)\right)$ for $0 \le x \le 2L$; $y_0 = 0$ otherwise, as shown in the Figure. Determine the motion of the center of mass of the car while passing over the bump. Represent the car as a mass m attached to a massless spring of relaxed length ℓ_0 and spring constant k. Ignore friction and assume that the spring is vertical at all times.



B2 A planet of mass m is orbiting a star of mass M. The planet experiences a small drag force $\mathbf{F} = -\alpha \mathbf{v}$ due to its motion through the star's dense atmosphere. Assuming an essentially circular orbit with radius $r = r_0$ at $t = 0$, calculate the time dependence of the radius of the orbit.

B3 A gun is fired straight up. Assuming that the air drag on the bullet varies quadratically with speed as cv^2 (c is a known constant), and the bullet's mass is m,

a. write down the differential equation for the speed as a function of height $v(x)$, separately for the upward and downward motions;

b. solve the equation for the upward motion with the initial condition $v(x=0) = v_0$ and obtain the maximum height $x_{\max}$ reached by the bullet (hint: introduce the variable $u = v^2$);

c. solve the equation for the downward motion with the initial condition $v(x_{\max}) = 0$ and obtain the speed at the ground $v(0)$. Interpret your answer in terms of the energy conservation.

B4 A billiard ball of radius a sits on the surface of a billiard table. It is initially spinning about a horizontal axis with angular speed ω_0 and with zero linear speed. The coefficient of sliding friction between the ball and the billiard table is μ.

a. Find the time the ball travels before slipping ceases to occur.

b. What is the linear speed reached by the ball in that time?

c. What is the distance traveled by the ball in that time?

Physical Constants

speed of light .... $c = 2.998\times10^{8}$ m/s

Planck's constant .... $h = 6.626\times10^{-34}$ J·s

Planck's constant / 2π.... $\hbar = 1.055\times10^{-34}$ J·s

Boltzmann constant ...... $k_B = 1.381\times10^{-23}$ J/K

elementary charge .... $e = 1.602\times10^{-19}$ C

electric permittivity ...... $\varepsilon_0 = 8.854\times10^{-12}$ F/m

magnetic permeability ... $\mu_0 = 1.257\times10^{-6}$ H/m

molar gas constant.......... $R = 8.314$ J / mol·K

Avogrado constant .... $N_A = 6.022\times10^{23}$ mol^{-1}

electrostatic constant ... $k = (4\pi\varepsilon_0)^{-1} = 8.988\times10^{9}$ m/F

electron mass .... $m_{el} = 9.109\times10^{-31}$ kg

electron rest energy...... 511.0 keV

Compton wavelength .. $\lambda_C = h / m_{el}c = 2.426$ pm

proton mass .... $m_p = 1.673\times10^{-27}$ kg $= 1836 m_{el}$

1 bohr .... $a_0 = \hbar^2 / ke^2 m_{el} = 0.5292$ Å

1 hartree (= 2 rydberg) ... $E_h = \hbar^2 / m_{el} a_0^2 = 27.21$ eV

gravitational constant ... $G = 6.674\times10^{-11}$ m^3/ kg s^2

hc .... $hc = 1240$ eV·nm

Equations That May Be Helpful

<u>MECHANICS</u>

Properties of the Earth:

- Mass: $M = 5.98\times10^{24}$ kg
- Radius: $R = 6.38\times10^{6}$ m
- Gravitational acceleration at surface: $g = 9.81$ m/s^2

Gauss's Law for gravity

$$\oint_S \mathbf{g}\cdot d\mathbf{A} = -4\pi G m_{\text{encl.}}$$

General solution for an undamped driven oscillator:

$$x(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F}{m(\omega_0^2 - \omega^2)}\cos(\omega t)$$

THERMODYNAMICS

Clausius' theorem: $\sum_{i=1}^{N} \frac{Q_i}{T_i} \leq 0$, which becomes $\sum_{i=1}^{N} \frac{Q_i}{T_i} = 0$ for a reversible cyclic process of N steps.

For adiabatic processes in an ideal gas with constant heat capacity, $pV^{\gamma} = \text{const.}$

$$\frac{dp}{dT} = \frac{\lambda}{T \Delta V}$$

specific heat of water: 4186 J/(kg·K)
latent heat of ice melting: 334 J/g

$$C_V = \left(\frac{\delta Q}{dT}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V \qquad C_p = \left(\frac{\delta Q}{dT}\right)_p = T\left(\frac{\partial S}{\partial T}\right)_p \qquad TdS = C_V dT + T\left(\frac{\partial S}{\partial V}\right)_T dV$$

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial p}{\partial S}\right)_V$$
$$\left(\frac{\partial T}{\partial p}\right)_S = \left(\frac{\partial V}{\partial S}\right)_p$$
$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V$$
$$\left(\frac{\partial S}{\partial p}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_p$$

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence : $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

Curl : $\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$

Laplacian : $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$

Divergence : $\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$

Curl :

$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence : $\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$

Curl : $\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$

Laplacian : $\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

INTEGRALS

$f(x)$	$\int_0^\infty f(x)\,dx$
e^{-ax^2}	$\frac{\sqrt{\pi}}{2\sqrt{a}}$
xe^{-ax^2}	$\frac{1}{2a}$
$x^2e^{-ax^2}$	$\frac{\sqrt{\pi}}{4a^{3/2}}$
$x^3e^{-ax^2}$	$\frac{1}{2a^2}$
$x^4e^{-ax^2}$	$\frac{3\sqrt{\pi}}{8a^{5/2}}$
$x^5e^{-ax^2}$	$\frac{1}{a^3}$
$x^6e^{-ax^2}$	$\frac{15\sqrt{\pi}}{16a^{7/2}}$

$$\int_0^\infty \frac{1}{1+ay^2}dy = \pi/2a^{1/2}$$

$$\int_0^\infty y^n e^{-y/a}dy = n!a^{n+1}$$

$$\int \frac{r^3dr}{(x^2+r^2)^{3/2}} = (r^2+x^2)^{1/2} + \frac{x^2}{(r^2+x^2)^{1/2}}$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) \qquad \left|\tan^{-1}\left(\frac{x}{a}\right)\right| < \frac{\pi}{2}$$

$$\int \frac{xdx}{a^2+x^2} = \frac{1}{2}\ln\left(a^2+x^2\right)$$

$$\int \frac{dx}{x(a^2+x^2)} = \frac{1}{2a^2}\ln\left(\frac{x^2}{a^2+x^2}\right)$$

$$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right)$$

$$= -\frac{1}{ab}\coth^{-1}\left(\frac{ax}{b}\right)$$

$$= -\frac{1}{ab}\tanh^{-1}\left(\frac{ax}{b}\right), \quad a^2x^2 < b^2$$