QM A1

Method (1)
Use

$$[x,p] = xp - px = i\hbar \quad \Rightarrow$$
$$xp = px + i\hbar \quad \text{and} \quad px = xp - i\hbar$$

Then

$$[xp,px] = xppx - px(xp) = xppx - px(px + ih) = xppx - (px)px - pxih =$$
$$= xppx - (xp - ih)px - pxih = xppx - xppx + pxih - pxih = 0$$

Method (2)

$$-\hbar^2(xppx)f = x(\ (xf)'\ )' = x(f + xf')' = x(f' + f' + xf'') = 2xf' + x^2 f''$$
$$-\hbar^2(pxxp)f = (\ (x^2 f')\ )' = 2xf' + x^2 f''$$

We see that $(xppx)f = (pxxp)f \quad \Rightarrow \quad xppx = pxxp \quad \Rightarrow \quad [xp,px] = 0$

Method (3)

Use $px = xp - i\hbar$, and then:

$$[xp,px] = [xp, xp - i\hbar] = [xp,xp] - [xp, i\hbar] = 0 - 0 = 0$$

QM A2

Part a.

One requirement for linearity is $\mathsf{A}(f+g)=\mathsf{A}f+\mathsf{A}g$.

Here, we have

$$\mathsf{A}(f+g)=\left|f+g\right|$$
$$\mathsf{A}f+\mathsf{A}g=\left|f\right|+\left|g\right|$$

but these two are not equal for all functions f and g, so the operator is NOT linear.
Counterexample: consider the constant functions $f(x)=1$ and $g(x)=-1$.

$$\mathsf{A}(f+g)=\left|1+(-1)\right|=0$$
$$\mathsf{A}f+\mathsf{A}g=\left|1\right|+\left|-1\right|=2$$

Part b.

The adjoint $\mathsf{O}^{\dagger}$ *of operator* O *is defined by* $\langle f\,|\,\mathsf{O}^{\dagger}g\rangle=\langle \mathsf{O}f\,|\,g\rangle$ *. Therefore, if some operator* O *is self-adjoint (* $\mathsf{O}^{\dagger}=\mathsf{O}$ *), it must satisfy* $\langle f\,|\,\mathsf{O}g\rangle=\langle \mathsf{O}f\,|\,g\rangle$ *.*

Checking this requirement for operator B we note it doesn't satisfy this requirement:

$\langle f\,|\,\mathsf{B}g\rangle=\langle f\,|\,g^*\rangle$, but

$\langle \mathsf{B}f\,|\,g\rangle=\langle f^*\,|\,g\rangle=\langle g\,|\,f^*\rangle^*=\langle g^*\,|\,f\rangle=\langle f\,|\,\mathsf{B}g\rangle^*\neq\langle f\,|\,\mathsf{B}g\rangle$

The two are not equal for all functions f and g, so the operator is NOT self-adjoint.

QM A3

A krypton fluoride (KrF) laser beam (wavelength $\lambda = 248$ nm) hits a polished metal surface. As a result, electrons leave the metal, the fastest ones having a speed of 7.96×10^5 m/s .

a. Find the energy per photon in the KrF laser beam, in eV.
b. Find the work function Φ of the metal, in eV.

ANSWERS

Part a.

$$\frac{E}{e} = \frac{hc}{\lambda e} = 5.00 \text{ eV}$$

```
In[200]:= mel = ElectronMass
          h = PlanckConstant
          cc = SpeedOfLight
          ee = ElectronCharge
Out[200]= 9.10938 × 10^-31 Kilogram
Out[201]= 6.62607 × 10^-34 Joule Second
Out[202]= 299 792 458 Meter / Second
Out[203]= 1.60218 × 10^-19 Coulomb
In[204]:= h * cc / (248*^-9 * ee)
Out[204]= 4.99936 Joule Meter / Coulomb
```

```
In[205]:= v = 7.96*^5
Out[205]= 796 000.
In[206]:= (1/2) mel v^2 / ee
Out[206]= 1.80125 Kilogram / Coulomb
```

Part b.

$$\frac{K}{e} = \frac{\frac{1}{2} m_{\text{el}} v^2}{e} = 1.80 \text{ eV}$$

Now $K = E - \Phi \quad \Rightarrow \quad \Phi = E - K = 5.00 - 1.80 = 3.20$ eV

Quantum A4

4.

(a) yes, $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = \frac{\hbar^2 k^2}{2m}\psi$

(b) no, $-i\hbar\frac{d}{dx}\psi \neq \text{const.}\,\psi$

(c) $\psi(x)$ is a superposition of states with $p=\hbar k$ and $p=-\hbar k$ each has 50% probability

QM B1

Part a.

First, we note $S_x^\dagger = S_x$ and $S_y^\dagger = S_y$, because spin components are observables. From this we obtain $S_+^\dagger = (S_x + iS_y)^\dagger = S_x^\dagger + (iS_y)^\dagger = S_x - iS_y = S_-$; hence, also $S_-^\dagger = S_+$. Next, we note that $(S_+S_-)^\dagger = S_-^\dagger S_+^\dagger = S_+S_-$, so the operator S_+S_- is self-adjoint.

Alternative solution:

$$S_+S_- = (S_x + iS_y)(S_x - iS_y) = S_x(S_x - iS_y) + iS_y(S_x - iS_y) = S_xS_x - iS_xS_y + iS_yS_x + S_yS_y =$$
$$= S_x^2 + S_y^2 - i[S_x, S_y] = S_x^2 + S_y^2 - i(i\hbar S_z) = S_x^2 + S_y^2 + \hbar S_z$$

which, being a sum of self-adjoint operators (observables), is self-adjoint.

Part b.

From $S_\pm = S_x \pm iS_y$ we find

$$S_x = \tfrac{1}{2}(S_+ + S_-) = \tfrac{1}{2}\hbar\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} = \tfrac{1}{2}\hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$S_y = -\tfrac{1}{2}i(S_+ - S_-) = -\tfrac{1}{2}i\hbar\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} = -\tfrac{1}{2}i\hbar \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \tfrac{1}{2}\hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Some students may double-check these matrices for self-adjointness:

$$S_x^\dagger = \tfrac{1}{2}\hbar\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\mathrm{T}} \right\}^* = \tfrac{1}{2}\hbar\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}^* = \tfrac{1}{2}\hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = S_x$$
$$S_y^\dagger = \tfrac{1}{2}\hbar\left\{ \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^{\mathrm{T}} \right\}^* = \tfrac{1}{2}\hbar\left\{ \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \right\}^* = \tfrac{1}{2}\hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = S_y$$

Part c.

$$S_y\psi = (+\tfrac{1}{2})\hbar\psi \quad ; \quad \text{take } \psi = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow$$

$$\cancel{\tfrac{1}{2}\hbar} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \cancel{\tfrac{1}{2}\hbar}\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow -ib = a \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -ib \\ b \end{pmatrix} \propto \begin{pmatrix} -i \\ 1 \end{pmatrix}, \text{ normalized: } \tfrac{1}{2}\sqrt{2}\begin{pmatrix} -i \\ 1 \end{pmatrix}$$

Check: $\tfrac{1}{2}\hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\left[\tfrac{1}{2}\sqrt{2}\begin{pmatrix} -i \\ 1 \end{pmatrix}\right] = \tfrac{1}{2}\hbar\tfrac{1}{2}\sqrt{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} -i \\ 1 \end{pmatrix} = \tfrac{1}{2}\hbar\left[\tfrac{1}{2}\sqrt{2}\begin{pmatrix} -i \\ 1 \end{pmatrix}\right]$

Part d.

This probability is $\left|\frac{1}{2}\sqrt{2}\left(0,1\right)^{*}\circ\begin{pmatrix}-i\\1\end{pmatrix}\right|^{2}=\frac{1}{2}\left|1\right|^{2}=\frac{1}{2}=50\%$.

Part e.

The spin ket collapsed to $\begin{pmatrix}0\\1\end{pmatrix}$, so this probability is $\left|\left(1,0\right)^{*}\circ\begin{pmatrix}0\\1\end{pmatrix}\right|^{2}=0$.

QM B2

Part a.

We have $\psi(x,0)=\sum_n c_n\varphi_n$, with (ignoring the common factor A) "unnormalized" expansion coefficients $c_0=2,\ c_1=-\frac{3}{2}\sqrt{2},\ c_2=\sqrt{2},\ c_{n>2}=0$.

Now $\sum_n |c_n|^2=|2|^2+\left|-\frac{3}{2}\sqrt{2}\right|^2+\left|\sqrt{2}\right|^2=4+\frac{9}{2}+2=\frac{21}{2}$. The probabilities P_n to find energy $E_n=(n+\frac{1}{2})\hbar\omega$ are:

$$P_0=\frac{|c_0|^2}{|c_0|^2+|c_1|^2+|c_2|^2}=\frac{4}{\frac{21}{2}}=\frac{8}{21}$$
$$P_1=\frac{|c_1|^2}{|c_0|^2+|c_1|^2+|c_2|^2}=\frac{\frac{9}{2}}{\frac{21}{2}}=\frac{9}{21}$$
$$P_2=\frac{|c_2|^2}{|c_0|^2+|c_1|^2+|c_2|^2}=\frac{2}{\frac{21}{2}}=\frac{4}{21}$$

(check: $P_0+P_1+P_2=\frac{8}{21}+\frac{9}{21}+\frac{4}{21}=1$; correct)

and so

$$\langle E\rangle=P_0E_0+P_1E_1+P_2E_2=\frac{8}{21}\tfrac{1}{2}(\hbar\omega)+\frac{9}{21}\tfrac{3}{2}(\hbar\omega)+\frac{4}{21}\tfrac{5}{2}(\hbar\omega)=\left(\frac{8}{42}+\frac{27}{42}+\frac{20}{42}\right)(\hbar\omega)=\frac{55}{42}\hbar\omega\left(\approx 1.31\hbar\omega\right)$$

Part b.

The stationary states are also parity eigenstates: $\hat{P}\varphi_n=\eta_n\varphi_n=(-1)^n\varphi_n$. So

$$\langle\hat{P}\rangle=\sum_n P_n\eta_n=\sum_n P_n(-1)^n=P_0-P_1+P_2=\frac{8}{21}-\frac{9}{21}+\frac{4}{21}=\frac{3}{21}=\frac{1}{7}$$

Part c.

The time evolution may be described as follows:

$$\psi(x,0)=A\left[(2)\varphi_0+\left(-\tfrac{3}{2}\sqrt{2}\right)\varphi_1+\left(\sqrt{2}\right)\varphi_2\right]\xrightarrow[\text{elapses}]{\text{time }T}\psi(x,T)=B\left[(2)\varphi_0+\left(-\tfrac{3}{2}\boxed{i}\sqrt{2}\right)\varphi_1+\left(\boxed{-1}\sqrt{2}\right)\varphi_2\right]$$

in which the numbers in boxes are the phase factors by which the individual expansion coefficients change as a result of this time evolution (relative to the 0-th expansion coefficient).

A phase factor potentially accumulated in the 0-th expansion coefficient may have been absorbed in the constant B.

From the general time-evolution of the expansion coefficients,

$$c_n(T) = c_n(0)e^{-iE_nT/\hbar} = c_n(0)e^{-i(n+1/2)\hbar\omega_0 T/\hbar} = c_n(0)e^{-i(n+1/2)\omega_0 T}$$

we now deduce:

$$\frac{c_2(T)}{c_0(T)} = \frac{c_2(0)e^{-(5/2)i\omega_0 T}}{c_0(0)e^{-i(1/2)\omega_0 T}} = \frac{c_2(0)}{c_0(0)}e^{-2i\omega_0 T} \quad \Rightarrow \quad \frac{\boxed{-1}\sqrt{2}}{2} = \frac{\sqrt{2}}{2}e^{-2i\omega_0 T} \quad \Rightarrow \quad e^{-2i\omega_0 T} = -1 \qquad \text{and}$$

$$\frac{c_2(T)}{c_1(T)} = \frac{c_2(0)e^{-(5/2)i\omega_0 T}}{c_1(0)e^{-(3/2)\omega_0 T}} = \frac{c_2(0)}{c_1(0)}e^{-i\omega_0 T} \quad \Rightarrow \quad \frac{\boxed{-1}\sqrt{2}}{-\frac{3}{2}\boxed{i}\sqrt{2}} = \frac{\sqrt{2}}{-\frac{3}{2}\sqrt{2}}e^{-i\omega_0 T} \quad \Rightarrow \quad e^{-i\omega_0 T} = i$$

The ω_0 angular frequency is slower than the $2\omega_0$ frequency, so we find the earliest time for the former frequency:

$$e^{-i\omega_0 T} = i = e^{-i(\frac{3}{2}\pi + N2\pi)} \quad \Rightarrow \quad \omega_0 T = \tfrac{3}{2}\pi + \cancel{N2\pi} \quad \Rightarrow \quad T = \frac{\frac{3}{2}\pi}{\omega_0}$$

QM B3

Parts a. and b.

Excluding the origin, the TISE is

$$\psi'' = \frac{2m|E|}{\hbar^2}\psi = \kappa^2\psi \quad \Rightarrow \quad \psi \propto e^{\pm\kappa x}$$

The wavefunction must have zero asymptotes at $x = \pm\infty$, so we set

$$\psi(x) = \begin{cases} \psi_{\rm L}(x) = Ae^{\kappa x} & x \le 0 \\ \psi_{\rm R}(x) = Be^{-\kappa x} & x \ge 0 \end{cases}$$

and start by requiring $\psi_{\rm L}(x=0) = \psi_{\rm R}(x=0) \quad \Rightarrow \quad A = B$, so we now have

$$\psi(x) = \begin{cases} \psi_{\rm L}(x) = Ae^{\kappa x} & x \le 0 \\ \psi_{\rm R}(x) = Ae^{-\kappa x} & x \ge 0 \end{cases}$$

The derivatives at the origin are

$$\psi'_{\rm L}(x=0) = \kappa A$$
$$\psi'_{\rm R}(x=0) = -\kappa A$$

With the given expression for the kink, $\lim_{\varepsilon \to 0}\left[\psi'(+\varepsilon) - \psi'(-\varepsilon)\right] = -\frac{2ma}{\hbar^2}\psi(0)$, this leads to

$$(-\kappa A) - (\kappa A) = -\frac{2ma}{\hbar^2}A \quad \Rightarrow \quad \kappa = \frac{ma}{\hbar^2}$$

(unit check: a is in J.m, so κ is in $\frac{\text{kg.J.m}}{\text{J}^2\text{s}^2} = \frac{\text{kg.m}}{\text{J.s}^2}\frac{\text{m}}{\text{m}} = \frac{\text{kg.m}^2}{\text{J.s}^2}\frac{1}{\text{m}} = \text{m}^{-1}$, correct)

We can now find the energy of the ground state:

$$|E| = \frac{\hbar^2\kappa^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{ma}{\hbar^2}\right)^2 = \frac{ma^2}{2\hbar^2}$$

Next, we normalize the wavefunction:

$$|\psi(x)|^2 = \begin{cases} \psi_{\mathrm{L}}{}^2(x) = A^2 e^{2\kappa x} & x \le 0 \\ \psi_{\mathrm{R}}(x) = A^2 e^{-2\kappa x} & x \ge 0 \end{cases}$$

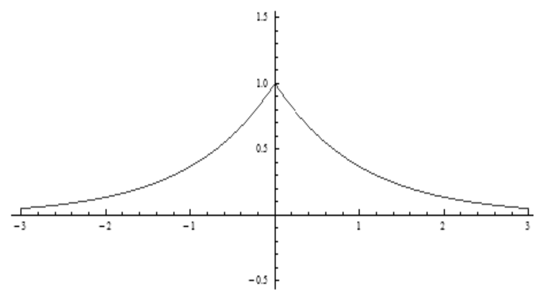
$$\int_{-\infty}^{\infty} |\psi(x)|^2 = A^2 \int_{-\infty}^{0} e^{2\kappa x} dx + A^2 \int_{0}^{\infty} e^{-2\kappa x} dx = 2A^2 \int_{0}^{\infty} e^{-2\kappa x} dx = 2A^2 \left[\frac{e^{-2\kappa x}}{-2\kappa} \right]_{x=0}^{x=\infty} =$$

$$= 2A^2 \left[\frac{e^{-2\kappa x}}{2\kappa} \right]_{x=\infty}^{x=0} = 2A^2 \frac{1}{2\kappa} = 1 \quad \Rightarrow \quad A = \sqrt{\kappa}$$

So ground state is $\psi(x) = \begin{cases} \psi_{\mathrm{L}}(x) = \sqrt{\kappa}\, e^{\kappa x} & x \le 0 \\ \psi_{\mathrm{R}}(x) = \sqrt{\kappa}\, e^{-\kappa x} & x \ge 0 \end{cases}$ and its energy is $E = -|E| = -\frac{ma^2}{2\hbar^2}$

(Check units: a is in J.m, so $|E|$ is in

$\frac{\mathrm{kg} \cdot \not{\mathrm{J}^2} \cdot \mathrm{m}^2}{\not{\mathrm{J}^2} \cdot \mathrm{s}^2} = \frac{\mathrm{kg} \cdot \mathrm{m}^2}{\mathrm{s}^2} = \mathrm{J}$, which is correct.)



Part c.

$\langle x \rangle = \int_{-\infty}^{\infty} \psi(x\psi) dx = 0$ (odd integrand)

Part d.

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} \psi(x^2\psi) dx = \int_{-\infty}^{\infty} x^2 \psi^2 dx = 2\kappa \left\{ \int_{0}^{\infty} x^2 e^{-2\kappa x} dx \right\} = 2\kappa \left\{ \frac{1}{4\kappa^3} \right\} = \frac{1}{2\kappa^2} = \frac{1}{2} \left(\frac{\hbar^2}{ma} \right)^2 = \frac{\hbar^4}{2m^2 a^2}$$

(the definite integral is in cheat sheet)

(Check units: a is in J.m, so $\langle x^2 \rangle = \frac{\hbar^4}{2m^2a^2}$ is in $\frac{\mathrm{J}^4 \mathrm{s}^4}{\mathrm{kg}^2 \mathrm{J}^2 \mathrm{m}^2} = \frac{\mathrm{J}^2 \mathrm{s}^4}{\mathrm{kg}^2 \mathrm{m}^2} = \frac{\mathrm{J}^2 \mathrm{s}^4 \mathrm{m}^2}{\mathrm{kg}^2 \mathrm{m}^4} = \mathrm{m}^2$, which is correct.)

Part e.

$$(\Delta x)^2 = \langle (x - \langle x \rangle)^2 \rangle = \langle x^2 - x\langle x \rangle - \langle x \rangle x + \langle x \rangle^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2 = \frac{\hbar^4}{2m^2 a^2}$$

Part f.

$\langle p \rangle = \frac{\hbar}{i} \int_{-\infty}^{\infty} \psi(\psi') dx$, and the integrand is odd, so $\langle p \rangle = 0$.

Or use integration by parts: $\langle p\rangle = \frac{\hbar}{i}\int_{-\infty}^{\infty}\psi(\psi')dx = \frac{\hbar}{i}\left[\psi^2 dx\right]_{x=-\infty}^{\infty} - \frac{\hbar}{i}\int_{-\infty}^{\infty}(\psi')\psi\, dx = 0 - \langle p\rangle \quad \Rightarrow \quad \langle p\rangle = 0$

Part g.

$$\langle p^2\rangle = \int_{-\infty}^{\infty}\psi(-\hbar^2\psi'')dx = 2m\int_{-\infty}^{\infty}\left[E\psi^2(x) + a\delta(x)\psi^2(x)\right]dx = 2m\left[E + a\psi^2(0)\right] =$$

$$2m\left[E + a\kappa\right] = 2m\left[-\frac{ma^2}{2\hbar^2} + a\frac{ma}{\hbar^2}\right] = \left[-\frac{m^2a^2}{\hbar^2} + \frac{2m^2a^2}{\hbar^2}\right] = \frac{m^2a^2}{\hbar^2}$$

Check units: $\langle p^2\rangle = \frac{m^2a^2}{\hbar^2}$ is in $\frac{\mathrm{kg}^2\mathrm{J}^2\mathrm{m}^2}{\mathrm{J}^2\mathrm{s}^2} = \frac{\mathrm{kg}^2\mathrm{m}^2}{\mathrm{s}^2} = \left(\frac{\mathrm{kg}\cdot\mathrm{m}}{\mathrm{s}}\right)^2$, correct.

Part h.

$$(\Delta p)^2 = \langle p^2\rangle - \langle p\rangle^2 = \frac{m^2a^2}{\hbar^2}$$

Part i.

$$(\Delta x)^2(\Delta p)^2 = \frac{\hbar^4}{2m^2a^2}\frac{m^2a^2}{\hbar^2} = \frac{\hbar^2}{2} \quad \Rightarrow \quad (\Delta x)(\Delta p) = \tfrac{1}{2}\sqrt{2}\,\hbar \sim \hbar$$

Quantum B4

(a)

$$[L_y, L_z] = i\hbar L_x$$

$$i\hbar \langle L_x \rangle = \langle L_y L_z \rangle - \langle L_z L_y \rangle$$

$$= \hbar m \langle L_y \rangle - \hbar m \langle L_y \rangle = 0$$

where we used $L_z |\psi\rangle = \hbar m |\psi\rangle$

and the hermiicity of the operator L_z

Similar $\langle L_y \rangle = 0$ from $[L_z, L_x] = i\hbar L_y$

(b) suppose spherical and azimuthal angle of z' axis are α and β. Then

$$\vec{L} \cdot \vec{e}_{z'} = L_x \sin\alpha \cos\beta + L_y \sin\alpha \sin\beta + L_z \cos\alpha$$

$$\langle \vec{L} \cdot \vec{e}_{z'} \rangle = \langle L_z \rangle \cos\alpha = \hbar m \cos\alpha$$

Since $\langle L_x \rangle = \langle L_y \rangle = 0$

$$(\vec{L} \cdot \vec{e}_{z'})^2 = L_x^2 \sin^2\alpha \cos^2\beta + L_y^2 \sin^2\alpha \sin^2\beta + L_z^2 \cos^2\alpha$$

where all crossed terms disappear because

$$\langle L_x L_y + L_y L_x \rangle = 0, \quad \langle L_x \rangle = \langle L_y \rangle = 0$$

Now $\langle L_x^2 \rangle = \langle L_y^2 \rangle = \frac{1}{2}(\vec{L}^2 - L_z^2) = \frac{\hbar^2}{2}[l(l+1) - m^2]$

Therefore $\langle (\vec{L} \cdot \vec{e}_{z'})^2 \rangle = \frac{\hbar^2}{2}[l(l+1) - m^2] \sin^2\alpha + \frac{\hbar^2}{2} m^2 \cos^2\alpha$

E&M A1

Two forces act on the small sphere: gravitational $\mathbf{F}_g = m\mathbf{g}$ and electrostatic $\mathbf{F}_e = q\mathbf{E}$.

$$E = \frac{\sigma}{2\varepsilon_0} = \frac{2.50\times10^{-9}}{2*8.85\times10^{-12}} = 141.3 \text{ N/C, so } F_e = 141.3\cdot5.00\times10^{-8} = 7.06\times10^{-6} \text{ N}$$

$$F_g = mg = 0.002\times10^{-3}\cdot9.81 = 1.96\times10^{-5} \text{ N}$$

And so $\tan\theta = \frac{F_e}{F_g} = \frac{7.06\times10^{-6}}{1.96\times10^{-5}} = 0.360 \quad \Rightarrow \quad \theta = 19.8°$ $\left(\tan\theta = \frac{F_e}{F_g} = \frac{qE}{mg} = \frac{q\sigma/2\varepsilon_0}{mg}\right)$

E&M A2

$$V_1 = \frac{Q}{C}, \qquad V_2 = \frac{Q}{2C}$$

reaches equilibrium $V = \frac{Q+Q}{C+2C} = \frac{2Q}{3C}$

The charge stored on left capacitor.

$$Q_{left} = 2CV = 2C\,\frac{2Q}{3C} = \frac{4}{3}Q \quad \#$$

E&M A3

a) $\vec{\Phi}_B = \vec{A} \cdot \vec{B} = (0.1m)^2 \times 0.6T$

$= \boxed{6 \times 10^{-3}\,Wb}$

b) $|\mathcal{E}_{AV}| = \dfrac{\Delta \Phi_B}{\Delta t}$; $\Delta \Phi_B = |\mathcal{E}_{AV}| \Delta t$

$= I_{av} R \Delta t$

$= \dfrac{\Delta Q}{\Delta t} R \Delta t = R \Delta Q$

$\Delta Q = \dfrac{1}{R} \Delta \Phi_B = \dfrac{2 \times 6 \times 10^{-3}}{15\Omega} Tm^2$

$= \boxed{0.8\,mC}$

E&M A4

Solution

Use the principle of superposition: this is the sum of a conductor of radius a with the current flowing into the page, and a conductor of radius $a/2$ with current flowing out of the page.

The current density is

$$J = \frac{I}{\pi a^2 - \pi(a/2)^2} = \frac{4I}{3\pi a^2}. \quad (1)$$

Let's start with point A: for the contribution from the large cylinder draw an Amperian loop at an arbitrary radius inside the cylinder. Then $\oint B \cdot dl = \mu_0 I_{enc}$ yields $B 2\pi s = \mu_0 J \pi s^2$ and

$$B(s) = \frac{2\mu_0 I}{3\pi a^2} s\hat{\phi}. \quad (2)$$

At $s = 0$, $B = 0$. For the small cylinder, a loop around the circumference of the cylinder gives $B 2\pi(a/2) = \mu_0 J \pi (a/2)^2$ so that

$$B = \frac{\mu_0 I}{3\pi a}(-\hat{\phi}), \quad (3)$$

which is thus the total field at A.

For the contribution from the large cylinder to point B, take a loop around the circumference, That gives

$$B = \frac{2\mu_0 I}{3\pi a}\hat{\phi}. \quad (4)$$

The contribution from the small cylinder requires drawing an Amperian loop centered on the small cylinder's axis with radius $3a/2$. That yields

$$B = \frac{\mu_0 I}{9\pi a}(-\hat{\phi}), \quad (5)$$

so that the total field at point B is

$$B_{tot} = \frac{5\mu_0 I}{9\pi a}\hat{\phi}. \quad (6)$$

E&M B1

This solution is not in SI, most candidates will use SI.

Solution:

The rotating charge produces current density $\vec{j} = \rho\vec{v}_{\rm rot}$, where $\rho = 3Q/4\pi R^3$. One can view the rotating sphere as a collection of rings of radii a carrying currents

$$dI = j\,dS = \rho\,\Omega\,a\,dS,$$

where dS is an element of vertical cross section of the sphere. Magnetic field $dB_{\rm pole}$ created by current loop dI is

$$dB_{\rm pole} = \frac{2\pi\,a^2\,dI}{c\,b^3},$$

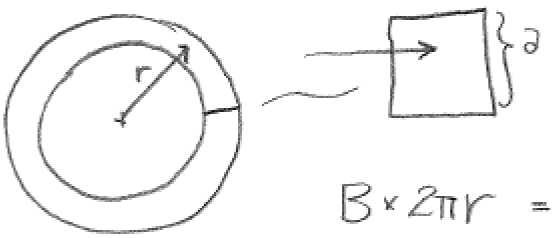
where b is the distance between the loop and the pole. Integrating over dI (half cross section of the sphere), one finds ($\sin\alpha \equiv a/b$)

$$B_{\rm pole} = \frac{2\pi\,\rho\,\Omega}{c}\int\frac{a^3}{b^3}\,dS = \frac{2\pi\,\rho\,\Omega}{c}\int\!\!\int\sin^3\alpha\;b\,d\alpha\,db = \frac{2\pi\,\rho\,\Omega}{c}\int_0^{2R}db\;b\int_{b/2R}^{1}d\cos\alpha\;\sin^2\alpha = \frac{2}{5}\frac{Q\Omega}{c\,R}.$$

[The magnetic dipole moment of the sphere is $\mu = (1/5)R^2\Omega\,Q/c$. One can show that the sphere produces dipole magnetic field at all $r \geq R$, so $B_{\rm pole} = 2\mu/R^3$.]

E&M B2

Assume $I = 1.5A$.



$N = 628$ turns

$$\int B\,dl = \mu_0 I_{encl}$$

$$B \times 2\pi r = \mu_0 N I$$

$$B = \frac{\mu_0 N I}{2\pi r}$$

$$\Phi_B = \int_{r_i}^{r_o} B a\,dx = \frac{\mu_0 N I}{2\pi}\int_{3cm}^{3.5cm} \frac{dx}{x}$$

$$= \frac{\mu_0 N I}{2\pi} \ln\left(\frac{35}{30}\right)$$

$$= \frac{4\pi \times 10^{-7} \times 628 \times 1.5}{2\pi} \ln(35/30)$$

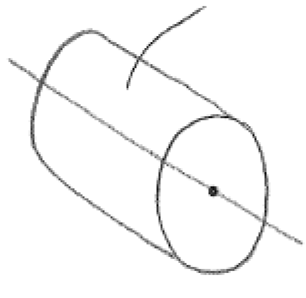
$$= 1.884 \times 10^{-4} \times 0.154 = 2.9 \times 10^{-5}$$

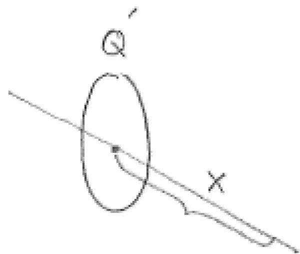
$$= \boxed{29\,\mu Wb}$$

TG (5/19/2015): In my solution for E&M B2, the answer I gave should have had a multiplicative factor of the height *a* (0.005 m)included.

E&M B3

$$\sigma = \frac{Q}{2\pi RL}$$





for a thin ring with total charge Q' & radius R, the field @ X (magnitude)

$$= \frac{kQx}{(x^2+R^2)^{3/2}}$$

(They should probably be able to derive this)

Thus

$$E = \int_{x=0}^{L} \frac{kx\,dQ}{(x^2+R^2)^{3/2}}$$

$$= \int_0^L \frac{kx\sigma\,dx\,2\pi R}{(x^2+R^2)^{3/2}}$$

$$= k[Q/(2\pi RL)]\,2\pi R * \int_0^L \frac{x\,dx}{(x^2+R^2)^{3/2}} =$$

$$\frac{kQ}{L}\frac{1}{\sqrt{x^2+R^2}}\Bigg|_L^0 = \frac{kQ}{L}\left(\frac{1}{R} - \frac{1}{\sqrt{R^2+L^2}}\right)$$

$$= \frac{kQ}{RL}\left(1 - \frac{1}{\sqrt{1+(L/R)^2}}\right)$$

E&M B4

Part a.

$V_R = IR, \quad V_L = IX_L = I\omega_d L$

Part b.

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + X_L{}^2} = \sqrt{R^2 + \omega_d^2 L^2}$$

Part c.

The power dissipated in the lightbulb is $P(L) = \frac{\mathcal{E}_{\text{rms}}^2}{Z} = \frac{\mathcal{E}_{\text{rms}}^2}{\sqrt{R^2 + \omega_d^2 L^2}}$.

The maximum power of 100 W is obtained for the minimum value of Z, so when $L \to 0$.

Part d.

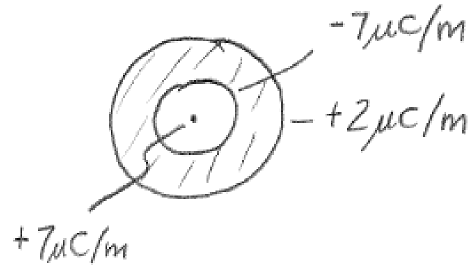
We require $P_{\text{avg}} = I_{\text{rms}}^2 R = \left(\frac{\mathcal{E}_{\text{rms}}}{Z}\right)^2 R = \frac{\mathcal{E}_{\text{rms}}^2}{R}\frac{R^2}{Z^2} = P_{\text{avg,max}}\frac{R^2}{Z^2} = \frac{30}{100} * P_{\text{avg,max}}$.

This implies

$$\frac{R^2}{Z^2} = \tfrac{3}{10} \quad \Rightarrow \quad Z^2 = R^2 + (\omega_d L)^2 = \tfrac{10}{3}R^2 \quad \Rightarrow \quad L = \sqrt{\tfrac{7}{3}}R/\omega_d = \sqrt{\tfrac{7}{3}}\frac{144}{2\pi * 60} = 0.58 \text{ H}.$$

Note that R may be calculated from $P = \frac{\mathcal{E}_{\text{rms}}^2}{R} = 100 \text{ W} \quad \Rightarrow \quad R = \frac{\mathcal{E}_{\text{rms}}^2}{P} = \frac{(120)^2}{100} = 144\,\Omega$.

E&M PREVIOUS A1



a) Charge along 1m of line

$= +2\frac{\mu C}{m} \times 1m = +2\mu C$

$\Rightarrow$ areal charge density at outer surface $= \frac{2\times10^{-6}C}{1m \times 2\pi(0.02m)}$

$= \frac{10^{-4}C}{2\pi m^2} = \boxed{1.59\times10^{-5}\ C/m^2}$

b) net line charge $= +2\mu C/m$

$E = \frac{2\lambda k}{r} = \frac{2\times8.99\times10^9\times2\times10^{-6}}{0.05m}$

$= \boxed{7.19\times10^5\ \frac{N}{C}}$

Step 1 - Apply Gauss' Law to a cylinder that is inside the conductor everywhere. Flux = 0 so net charge is 0. The +7 uC/m on the central line thus attracts –7 uC/m on the inner surface. Total charge of conductor is –5 uC/m, so on outer surface there is –5 –(–7) = +2 uC/m. The area of the outer surface for one running meter of the cylinder is A = 2 pi x (outer radius) x 1 = 0.1257 m2. The charge on this 1 meter is 2 uC, so sigma = 2E-6 / 0.1257 = 1.59E-5 C.

Step 2 - Again Gauss' law with surface a cylinder of radius r and take its length L.
Call charge per meter labda, this equals +2 uC/m
Then 2 pi r L E = L x (labda) / eps0 → E(r) = (labda) / (eps0 2pi r) = 35967 / r
→E(r = 0.05 m) = 35967 / 0.05 = 719 kN/C.
Because net charge is positive, field points away from cylinder, perpendicular to its symmetry axis.