UNL - Department of Physics and Astronomy

Preliminary Examination - Day 2
Friday, May 13, 2016

This test covers the topics of *Thermodynamics and Statistical Mechanics* (Topic 1) and *Mechanics* (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

Note: If you do more than two problems in a group, only the first two (in the order they appear in this handout) will be graded. For instance, if you do problems A1, A3, and A4, only A1 and A3 will be graded.

WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY

Thermodynamics and Statistical Mechanics Group A - Answer only two Group A questions

A1 An insurance company needs to know the probability $P(n)$ that the state of Calisota is hit by n tornadoes in a year. They find there is a probability distribution:

$$P(n) \propto \frac{\mu^n}{n!},$$

where $\mu = 2.5$.

a. What is the probability that there will be no tornadoes in Calisota next year?
b. What is the probability that Calisota will be hit by 4 or more tornadoes next year?
c. What is the average number of tornadoes per year in Calisota?

A2 The molar energy of a monatomic gas which obeys van der Waals' equation is given by

$$E = \frac{3}{2}RT - \frac{a}{V},$$

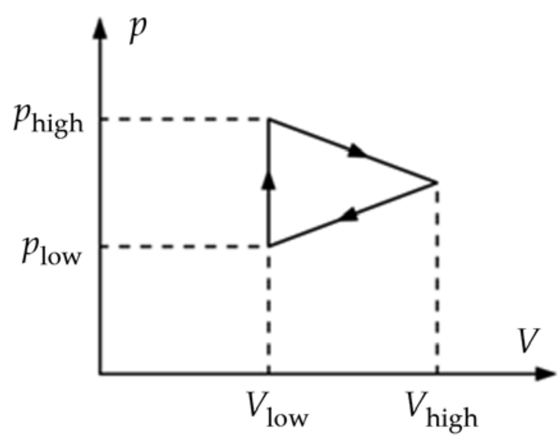
where V is the molar volume at temperature T and a is a constant. Initially, one mole of the gas is at temperature T_1 and occupies a volume V_1. The gas is then allowed to expand adiabatically into a vacuum so that it occupies a total volume V_2. What is the final temperature of the gas?

A3 Derive the Maxwell relation $\left(\frac{\partial p}{\partial T}\right)_V = \left(\frac{\partial S}{\partial V}\right)_T$. Hint: you can find a list of the full differentials for thermodynamic potentials in the formula sheet.

A4 The pV cycle of an engine is shown in the figure. Let

$p_{\text{low}} = 1 \times 10^6$ Pa ,
$p_{\text{high}} = 2 \times 10^6$ Pa ,
$V_{\text{low}} = 0.005$ m^3,
and $V_{\text{high}} = 0.025$ m^3.

If the energy absorbed by heating the engine is 50,000 J, what is the efficiency of the engine?

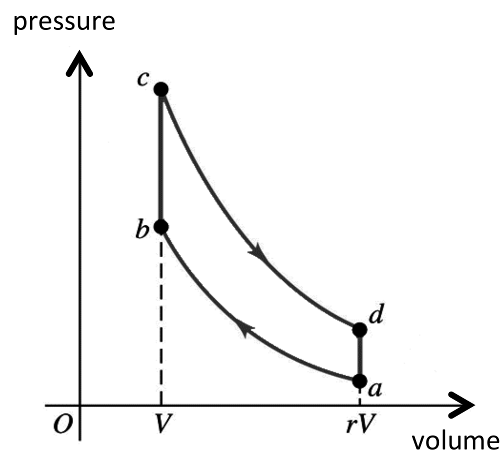


Thermodynamics and Statistical Mechanics Group B - Answer only two Group B questions

B1 The pV diagram shows the *Otto cycle,* which is an idealized model of the thermodynamic processes in a gasoline engine. Processes $a \to b$ and $c \to d$ are adiabatic, and processes $b \to c$ and $d \to a$ are isochoric (taking place at constant volume). The quantity $r > 1$ is called the compression ratio. Show that the efficiency η of the Otto cycle is

$$\eta = 1 - \frac{1}{r^{\gamma-1}},$$

where $\gamma = C_p / C_V$.



B2 Suppose there are two kinds of *E. coli* bacteria, "red" ones and "green" ones. Each reproduces by splitting in two: red $\to$ red + red or green $\to$ green + green, with a reproduction time of one hour. Other than the markers "red" and "green" there are no differences between them. A colony of 5,000 red and 5,000 green *E. coli* is allowed to eat and reproduce. In order to keep the colony size down, a predator is introduced which keeps the colony size at 10,000 by eating bacteria. The predator eats the bacteria randomly: there is a 50% probability that it eats a red one, and a 50% probability that it eats a green one. What is the probability distribution of the number of red bacteria after a very long time?

B3 The state equation of a new kind of matter is

$$p = \frac{AT^3}{V},$$

where p, V, and T are the pressure, volume, and temperature, respectively, while A is a constant. The internal energy of the matter is

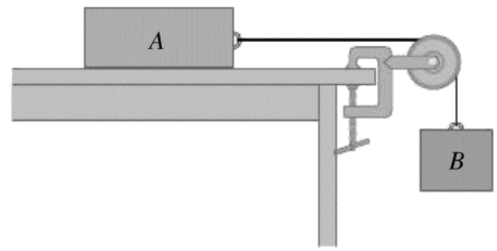
$$U = BT^n \ln\frac{V}{V_0} + f(T),$$

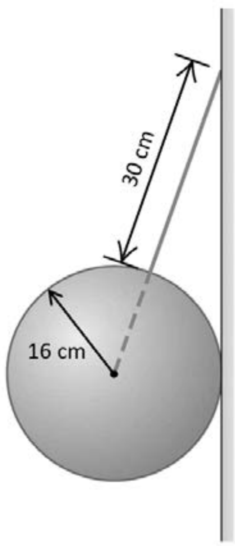
where $B, n,$ and V_0 are all constants, and $f(T)$ only depends on the temperature. Find B and n. Consider the other constants to be known.

B4 Using the equation of state $pV = nRT$ and the specific heat per mole $C_V = \frac{3}{2}R$ for a monatomic ideal gas, find its Helmholtz free energy as a function of the number of moles n, V, and T.

Mechanics Group A - Answer only two Group A questions

A1 Two blocks are connected by a massless cord which passes over a massless and frictionless pulley. Block A sits on a level, frictionless surface, and is sliding to the right. The masses of the blocks A and B are $m_A = 7\text{ kg}$ and $m_B = 3\text{ kg}$. Calculate the tension T in the cord.





A2 A solid, uniform, 45 kg ball of radius 16 cm is supported against a vertical, frictionless wall by a thin, 30 cm wire of negligible mass.

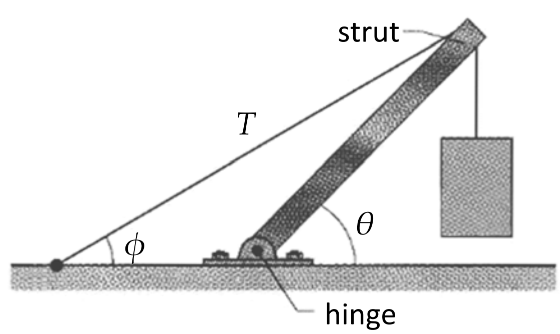
Find the tension in the wire and the magnitude of the force that the ball exerts on the wall.

A3 You lift a very flexible chain of linear mass density μ kg/m off of a table with your hand moving vertically upward with a speed of v m/s. What is the magnitude of the force the chain exerts on your hand as a function of your hand's height h above the table?

A4 The system in the figure is in equilibrium. A concrete block of mass 225 kg hangs from the end of a uniform strut of mass 45 kg and unknown length. The cable has negligible mass. The angles are $\phi = 30°$ and $\theta = 45°$.

a. Find the tension T in the cable.

b. Find the horizontal and vertical components of the force on the strut by the hinge.



Mechanics Group B - Answer only two Group B questions

B1 A large, uniform, solid, circular disk of radius R and mass M has a small hole drilled through it a distance r from its center. The disk acts as a physical pendulum when it pivots about a thin horizontal rod passing through this hole. What must r be ($0 < r < R$) to minimize the period of small-amplitude pendular motion?

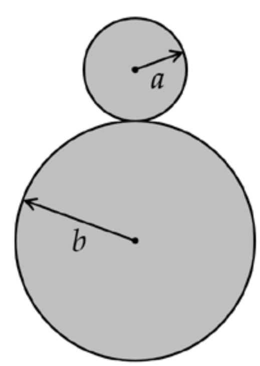
B2 A particle of mass m is performing one-dimensional motion subject to the force function

$$F(x) = -F_0 \sin(cx) .$$

At some instant the particle's position is $x = 0$ and its velocity $v = v_0$.

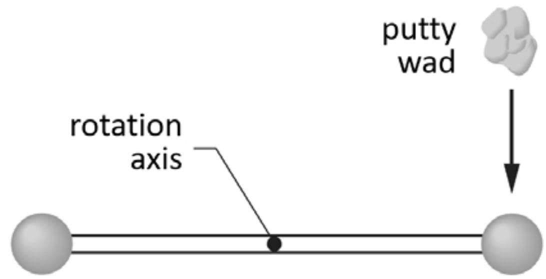
a. Find the potential energy as a function of x and sketch it.
b. Find the velocity as a function of x.
c. Find the condition on the initial velocity v_0 for which the motion is periodic ($v_0 <$?).
d. Find the turning points for the periodic motion.
e. Suppose that v_0 is so small that the periodic motion can be treated as harmonic oscillations. Find the period of these oscillations and the oscillation amplitude.

B3 A uniform sphere of radius a is balanced on top of a fixed cylinder of radius b ($b > a$). The balance is slightly disturbed so that the sphere starts rolling down without slipping in the plane perpendicular to the cylinder axis.



a. Show that the sphere leaves the cylinder when the line of centers makes an angle of $\cos^{-1}(10/17)$ with the vertical.
b. How will your answer change if the sphere is hollow? In which case is the angle bigger?

B4 Two balls, each of mass $m = 2.00$ kg and with negligible radius, are attached to a thin rod of length $L = 50.0$ cm of negligible mass. The rod is free to rotate in a vertical plane without friction about a horizontal axis through its center. With the rod initially stationary and horizontal, a wad of putty of mass $M = 50.0$ g drops onto one of the balls, hitting it with a speed $v_0 = 3.00$ m/s and then sticking to it.



- *a.* What is the angular speed of the system just after the putty wad hits?
- *b.* What is the ratio of the kinetic energy of the system after the collision to that of the putty wad just before?
- *c.* Through what angle will the system rotate before it momentarily stops?

Physical Constants

speed of light.................. $c = 2.998\times10^{8}$ m/s

Planck's constant .... $h = 6.626\times10^{-34}$ J·s

Planck's constant / 2π.... $\hbar = 1.055\times10^{-34}$ J·s

Boltzmann constant ...... $k_{\mathrm{B}} = 1.381\times10^{-23}$ J/K

elementary charge .... $e = 1.602\times10^{-19}$ C

electric permittivity ...... $\varepsilon_0 = 8.854\times10^{-12}$ F/m

magnetic permeability ... $\mu_0 = 1.257\times10^{-6}$ H/m

molar gas constant.......... $R = 8.314$ J / mol·K

Avogrado constant .... $N_{\mathrm{A}} = 6.022\times10^{23}$ mol^{-1}

electrostatic constant ... $k = (4\pi\varepsilon_0)^{-1} = 8.988\times10^{9}$ m/F

electron mass .... $m_{\mathrm{el}} = 9.109\times10^{-31}$ kg

electron rest energy...... 511.0 keV

Compton wavelength .. $\lambda_{\mathrm{C}} = h / m_{\mathrm{el}}c = 2.426$ pm

proton mass .... $m_{\mathrm{p}} = 1.673\times10^{-27}\,\mathrm{kg} = 1836 m_{\mathrm{el}}$

1 bohr............................ $a_0 = \hbar^2 / ke^2 m_{\mathrm{el}} = 0.5292$ Å

1 hartree (= 2 rydberg) ... $E_{\mathrm{h}} = \hbar^2 / m_{\mathrm{el}} a_0{}^2 = 27.21$ eV

gravitational constant ... $G = 6.674\times10^{-11}$ m^3 / kg s^2

hc .... $hc = 1240$ eV·nm

Equations That May Be Helpful

POWER SERIES

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \quad \left(|x| \le 1,\ x \ne -1\right)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \quad \left(|x| < 1\right)$$

THERMODYNAMICS

Clausius' theorem: $\displaystyle\sum_{i=1}^{N} \frac{Q_i}{T_i} \le 0$, which becomes $\displaystyle\sum_{i=1}^{N} \frac{Q_i}{T_i} = 0$ for a reversible cyclic process of N steps.

For adiabatic processes in an ideal gas with constant heat capacity, $pV^{\gamma} = \text{const.}$

$$dU = TdS - pdV \qquad dH \equiv d(U + pV)$$

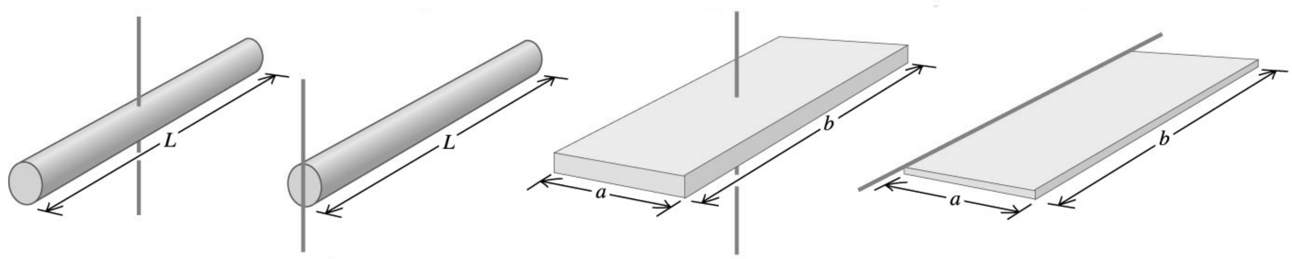
$$dF \equiv d(U - TS) \qquad dG = d(U + pV - TS)$$

MECHANICS

Gravitational acceleration at surface of Earth: $g = 9.81\ \text{m/s}^2$

Gauss's Law for gravity: $\oint_S \mathbf{g} \cdot d\mathbf{A} = -4\pi G\, m_{\text{encl.}}$

Moments of Inertia of Various Bodies

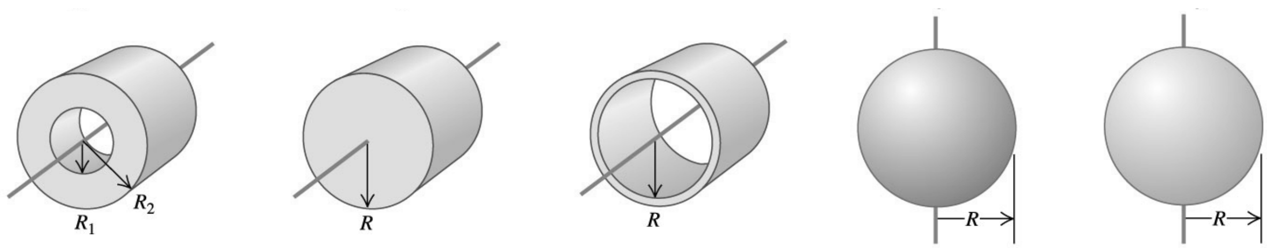


Slender rod, axis through center: $I = \frac{1}{12}ML^2$

Slender rod, axis through one end: $I = \frac{1}{3}ML^2$

Rectangular plate, axis through center: $I = \frac{1}{12}M\left(a^2 + b^2\right)$

Thin rectangular plate, axis along edge: $I = \frac{1}{3}Ma^2$



Hollow cylinder: $I = \frac{1}{2}M\left(R_1^2 + R_2^2\right)$

Solid cylinder: $I = \frac{1}{2}MR^2$

Thin-walled hollow cylinder: $I = MR^2$

Solid sphere: $I = \frac{2}{5}MR^2$

Thin-walled hollow sphere: $I = \frac{2}{3}MR^2$

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl : $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem : $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

INTEGRALS

$f(x)$		$\int_0^\infty f(x)\,dx$
e^{-ax^2}	....	$\frac{\sqrt{\pi}}{2\sqrt{a}}$
xe^{-ax^2}	....	$\frac{1}{2a}$
$x^2e^{-ax^2}$	....	$\frac{\sqrt{\pi}}{4a^{3/2}}$
$x^3e^{-ax^2}$	....	$\frac{1}{2a^2}$
$x^4e^{-ax^2}$	....	$\frac{3\sqrt{\pi}}{8a^{5/2}}$
$x^5e^{-ax^2}$	....	$\frac{1}{a^3}$
$x^6e^{-ax^2}$	....	$\frac{15\sqrt{\pi}}{16a^{7/2}}$

$$\int_0^\infty \frac{1}{1+ay^2}dy = \pi/2a^{1/2}$$

$$\int_0^\infty y^n e^{-y/a}dy = n!a^{n+1}$$

$$\begin{aligned}
\int x^{-2}\ln(x)\,dx &= -\frac{\ln(x)}{x}-\frac{1}{x}\\
\int x^{-1}\ln(x)\,dx &= \tfrac{1}{2}\ln^2(x)\\
\int \ln(x)\,dx &= x\ln(x)-x\\
\int x\ln(x)\,dx &= \tfrac{1}{2}x^2\ln(x)-\tfrac{1}{4}x^2\\
\int x^2\ln(x)\,dx &= \tfrac{1}{3}x^3\ln(x)-\tfrac{1}{9}x^3
\end{aligned}$$

$$\int \ln(1+x)dx = (x+1)\big(\ln(x+1)-1\big)+C$$

$$\begin{aligned}
\int\frac{r^3dr}{(x^2+r^2)^{3/2}} &= (r^2+x^2)^{1/2}+\frac{x^2}{(r^2+x^2)^{1/2}}\\
\int\frac{dx}{a^2+x^2} &= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) \qquad \left|\tan^{-1}\left(\frac{x}{a}\right)\right|<\frac{\pi}{2}\\
\int\frac{xdx}{a^2+x^2} &= \frac{1}{2}\ln\left(a^2+x^2\right)\\
\int\frac{dx}{x(a^2+x^2)} &= \frac{1}{2a^2}\ln\left(\frac{x^2}{a^2+x^2}\right)\\
\int\frac{dx}{a^2x^2-b^2} &= \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right)\\
&= -\frac{1}{ab}\coth^{-1}\left(\frac{ax}{b}\right)\\
&= -\frac{1}{ab}\tanh^{-1}\left(\frac{ax}{b}\right), \qquad a^2x^2<b^2
\end{aligned}$$