UNL - Department of Physics and Astronomy

Preliminary Examination - Day 1
Thursday, May 11, 2017

This test covers the topics of *Quantum Mechanics* (Topic 1) and *Electrodynamics* (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

Note: If you do more than two problems in a group, only the first two (in the order they appear in this handout) will be graded. For instance, if you do problems A1, A3, and A4, only A1 and A3 will be graded.

WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY

Quantum Mechanics Group A - Answer only two Group A questions

A1 Positronium is an atom consisting of electron and positron (positively-charged electron). Consider positronium in its ground state and find

(a) The most probable distance between the electron and the positron;

(b) The average distance between the electron and the positron.

Express both quantities in terms of the Bohr radius a_0.

A2 In a scattering experiment, protons with energy 2 eV are scattered from a crystal. The fifth maximum of the intensity is observed when the protons strike the crystal's surface at an angle of $30°$. Calculate the crystal's planar separation.

A3 A free particle is moving in one dimension. At $t = 0$ its wavefunction is given by

$$\psi(x) = A[\cos(kx) + \cos(kx/2)]$$

where A is a normalization constant such that

$$\int_{-L}^{L} |\psi(x)|^2 dx = 1,$$

where $2L$ can be considered as a large one-dimensional normalization "volume".

(a) Is this function an energy eigenstate?

(b) Is this function a parity eigenstate?

(c) Write down the particle's wavefunction at $t > 0$.

(d) Find the expectation values of momentum and parity at $t > 0$.

A4 An explicit form of the step-down ladder operator is

$$L_{-} = \hbar e^{-i\phi}\left(i\cot\theta \frac{\partial}{\partial\phi} - \frac{\partial}{\partial\theta} \right).$$

Apply this operator in this form to $Y_{2,-2}(\theta,\phi)$ and explain your result.

<u>Quantum Mechanics Group B</u> <u>- Answer only two Group B questions</u>

B1 At time $t = 0$, a hydrogen atom is in the superposition state

$$\psi(\mathbf{r}, 0) = \frac{1}{\sqrt{2}}[\psi_{100}(\mathbf{r}) + A(\psi_{210}(\mathbf{r}) + \psi_{211}(\mathbf{r}) + \psi_{21-1}(\mathbf{r}))]$$

where $\psi_{nlm}(\mathbf{r})$ is a normalized stationary state with the principal quantum number n, orbital angular momentum quantum number l and its projection m.

(a) Assuming $\psi(\mathbf{r}, 0)$ is normalized, and the constant A is real, find A;

(b) What is the radial probability density for this state?

(c) Find the expectation value of r for this state;

(d) At $t = 0$, L^2 is measured. What are the possible outcomes of this measurement and what are corresponding probabilities?

(e) At $t = 0$, L_z is measured. What are the possible outcomes of this measurement and what are corresponding probabilities?

(f) Given the initial state $\psi(\mathbf{r}, 0)$, find $\psi(\mathbf{r}, t)$.

(g) What is $\psi(\mathbf{r}, t)$, if at $t = 0$ measurement of L_z finds the value $\hbar$?

B2 Consider a system whose Hamiltonian is given by $\hat{H} = \alpha\left(|\varphi_1\rangle\langle\varphi_2| + |\varphi_2\rangle\langle\varphi_1|\right)$, where α is a real number, and $|\varphi_1\rangle$ and $|\varphi_2\rangle$ are normalized eigenkets of some Hermitian operator $\hat{A}$. Definition: An operator $\hat{\Omega}$ is said to be a projection operator if it is Hermitian and equal to its own square: $\hat{\Omega}^\dagger = \hat{\Omega}$ and $\hat{\Omega}^2 = \hat{\Omega}$.

a. Is $\hat{H}$ a projection operator? What about $\alpha^{-2}\hat{H}^2$?

b. Show that $|\varphi_1\rangle$ and $|\varphi_2\rangle$ are not eigenvectors of $\hat{H}$.

c. Calculate the commutators $[\hat{H}, |\varphi_1\rangle\langle\varphi_1|]$ and $[\hat{H}, |\varphi_2\rangle\langle\varphi_2|]$, and then find the relation between them.

d. Find the normalized eigenstates of $\hat{H}$ and their corresponding energy eigenvalues.

B3 Consider a particle whose wave function is

$$\psi(x,y,z) = \frac{1}{4\sqrt{\pi}} \frac{2z^2 - x^2 - y^2}{r^2} + \sqrt{\frac{3}{\pi}} \frac{xz}{r^2}.$$

a. Show that $\psi(x,y,z)$ is normalized.

b. Calculate $\langle \hat{L}^2 \rangle$ and $\langle \hat{L}_z \rangle$.

c. Calculate $\langle \hat{L}_+ \rangle$.

d. What is the probability of finding the particle in a small solid angle $\Delta\Omega = 10^{-3}$ sr in the direction $(\theta,\phi) = (\frac{1}{3}\pi, \frac{1}{2}\pi)$? Because $\Delta\Omega$ is so small, you may assume ψ is constant over $\Delta\Omega$.

B4 A spin-1/2 particle is in the state

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix}.$$

(a) Calculate expectation values of S_x, S_y, and S_z.

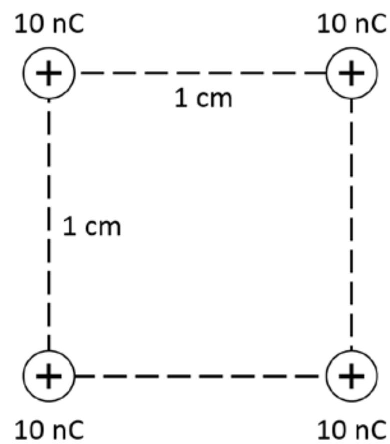
(b) Show that it is impossible to have a state such that $\langle S_x \rangle = \langle S_y \rangle = \langle S_z \rangle = 0$.

(c) Suppose the particle is in the state which is an eigenstate of S_x. What are the expectation values of S_y S_z, S_x^2 and S_z^2 for this state?

Electrodynamics Group A - Answer only two Group A questions

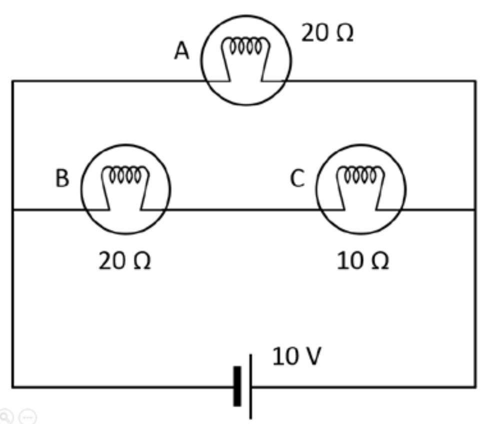
A1 An ideal square parallel-plate capacitor 5.0 cm on a side has a 0.50 mm gap. What is the total displacement current in the capacitor if the potential difference across the capacitor is increasing at 500 kV/s?

A2 The four 1 gram point masses in the figure are released simultaneously and allowed to move away from each other. What is the speed of each point mass when they are very far apart?



A3 The three lightbulbs A, B, and C in the circuit in the figure have resistance as indicated. The internal resistance of the battery is negligible.

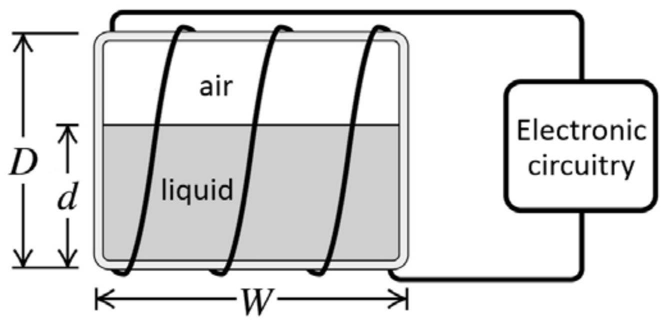
a. Calculate the current that passes through each lightbulb.
b. Calculate the voltage across each lightbulb.
c. Which lightbulb has the highest brightness (brightness $\propto$ power)?
d. What is the current passing through the battery?



A4 Consider two concentric spherical metal shells of radii r_1 and r_2 with $r_2 > r_1$. The outer shell has charge q and the inner shell is grounded. What is the charge on the inner shell?

Electrodynamics Group B - Answer only two Group B questions

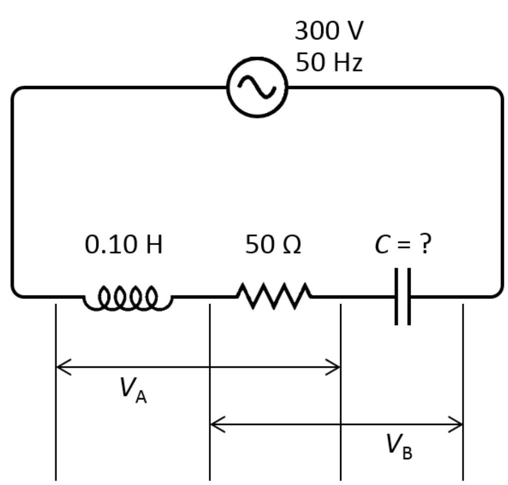
B1 A tank containing a liquid has turns of wire wrapped around it, causing it to act like an inductor. The liquid content of the tank can be measured by using its inductance to determine the height of the liquid in the tank. The inductance of the tank changes from a value of L_0, corresponding to a relative permeability of 1 when the tank is empty, to a value of L_f, corresponding to a relative permeability of μ_r (the relative permeability of the liquid) when the tank is full. The appropriate electronic circuitry can determine the inductance to five significant figures and thus the effective relative permeability of the combined air and liquid within the rectangular cavity of the tank. The four sides of the tank each have width W and height D. The height of the liquid in the tank is d. You can ignore any fringing effects and assume that the relative permeability of the material of which the tank is made and that of air can be ignored. The magnetic susceptibility of the fluid is $\chi_m = 6.4 \times 10^{-3}$.



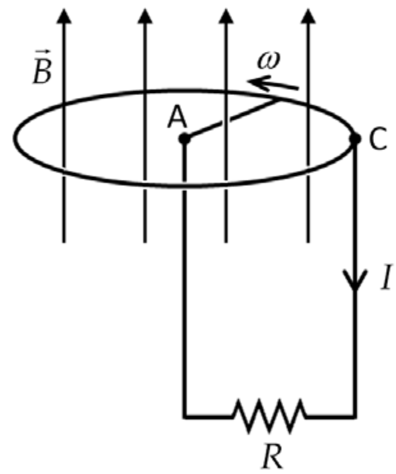
a. Derive an expression for d as a function of L (the measured inductance), L_0, L_f, and D.

b. What is the inductance (to five significant figures) for a tank $\frac{1}{4}$ full, $\frac{1}{2}$ full, $\frac{3}{4}$ full, and completely full?

B2 An inductor with inductance 0.10 H, a resistor with resistance 50 Ω, and a capacitor with an unknown capacitance are connected in series to a source of emf with frequency 50 Hz and potential amplitude 300 V. The ratio of the potential amplitudes in the two parts of the circuit indicated in the figure is $V_A / V_B = \frac{1}{2}$. Calculate the capacitance of the capacitor and the current amplitude in the circuit.

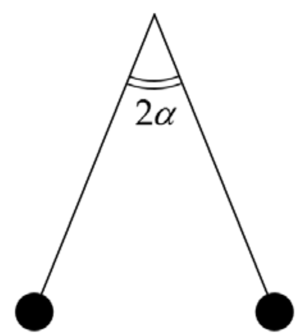


B3 One end of a conducting rod rotates with constant angular velocity ω in a circle of radius r making contact with a horizontal, conducting ring of the same radius. The other end of the rod is fixed. Stationary conducting wires connect the fixed end of the rod (A) and a fixed point on the ring (C) to either end of a resistance R. A uniform vertical magnetic field $\vec{B}$ passes through the ring.



a. Find the current I flowing through the resistor.
b. What is the sign of the current if positive I corresponds to flow in the direction of the arrow in the figure?
c. What torque must be applied to the rod to maintain its rotation at the constant angular velocity ω?

B4 Two identical balls, charged by the same electric charge, are hung by two equally long strings of negligible mass. The strings form an angle 2α, as shown in the figure. If this system is now immersed in benzene, the angle formed by the strings does not change. Find the density of the material the balls are made from.



The density of benzene is $\rho_b = 879\ \mathrm{kg \cdot m^{-3}}$, and its relative permittivity is $\varepsilon_r = 2.3$.

Physical Constants

speed of light .... $c = 2.998\times10^{8}$ m/s

Planck's constant .... $h = 6.626\times10^{-34}$ J·s

Planck's constant / 2π.... $\hbar = 1.055\times10^{-34}$ J·s

Boltzmann constant ...... $k_B = 1.381\times10^{-23}$ J/K

elementary charge .... $e = 1.602\times10^{-19}$ C

electric permittivity ...... $\varepsilon_0 = 8.854\times10^{-12}$ F/m

magnetic permeability ... $\mu_0 = 1.257\times10^{-6}$ H/m

molar gas constant.......... $R = 8.314$ J / mol·K

Avogadro constant .... $N_A = 6.022\times10^{23}$ mol^{-1}

electrostatic constant ... $k = (4\pi\varepsilon_0)^{-1} = 8.988\times10^{9}$ m/F

electron mass .... $m_{el} = 9.109\times10^{-31}$ kg

electron rest energy...... 511.0 keV

Compton wavelength .. $\lambda_C = h / m_{el}c = 2.426$ pm

proton mass .... $m_p = 1.673\times10^{-27}\,\mathrm{kg} = 1836 m_{el}$

1 bohr .... $a_0 = \hbar^2 / ke^2 m_{el} = 0.5292$ Å

1 hartree (= 2 rydberg) ... $E_h = \hbar^2 / m_{el}a_0^{\,2} = 27.21$ eV

gravitational constant ... $G = 6.674\times10^{-11}$ m^3 / kg s^2

hc .... $hc = 1240$ eV·nm

Equations That May Be Helpful

TRIGONOMETRY

$$\sin\alpha\sin\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$
$$\cos\alpha\cos\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$
$$\sin\alpha\cos\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$
$$\cos\alpha\sin\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$$

QUANTUM MECHANICS

Ground-state wavefunction of the hydrogen atom: $\psi(\mathbf{r}) = \dfrac{e^{-r/a_0}}{\pi^{1/2}a_0^{3/2}}$, where $a_0 = \dfrac{4\pi\varepsilon_0\hbar^2}{me^2}$ is the Bohr radius, using $m \approx m_{el}$, in which m_{el} is the electron mass.

$$\psi_{nlm}(\mathbf{r}) = R_{nl}(r)Y_{lm}(\hat{\mathbf{r}})$$
$$R_{10}(r) = \frac{2}{a_0^{3/2}}e^{-r/a_0}$$
$$R_{21}(r) = \frac{1}{3^{1/2}(2a_0)^{3/2}}\frac{r}{a_0}e^{-r/2a_0}$$
$$E_n = -\frac{1}{n^2}\frac{mk^2e^4}{2\hbar^2}$$

Particle in one-dimensional, infinitely-deep box with walls at $x=0$ and $x=a$:

Stationary states $\psi_n = (2/a)^{1/2}\sin(n\pi x/a)$, energy levels $E_n = n^2 \dfrac{\pi^2\hbar^2}{2ma^2}$

Angular momentum: $[L_x, L_y] = i\hbar L_z$ *et cycl.*

Ladder operators: $L_+ |\ell,m\rangle = \hbar\sqrt{(\ell+m+1)(\ell-m)}\,|\ell,m+1\rangle$

$L_- |\ell,m\rangle = \hbar\sqrt{(\ell+m)(\ell-m+1)}\,|\ell,m-1\rangle$

Table Spherical harmonics and their expressions in Cartesian coordinates.

$Y_{lm}(\theta,\varphi)$	$Y_{lm}(x,y,z)$
$Y_{00}(\theta,\varphi) = \frac{1}{\sqrt{4\pi}}$	$Y_{00}(x,y,z) = \frac{1}{\sqrt{4\pi}}$
$Y_{10}(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta$	$Y_{10}(x,y,z) = \sqrt{\frac{3}{4\pi}}\frac{z}{r}$
$Y_{1,\pm1}(\theta,\varphi) = \mp\sqrt{\frac{3}{8\pi}}e^{\pm i\varphi}\sin\theta$	$Y_{1,\pm1}(x,y,z) = \mp\sqrt{\frac{3}{8\pi}}\frac{x\pm iy}{r}$
$Y_{20}(\theta,\varphi) = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$	$Y_{20}(x,y,z) = \sqrt{\frac{5}{16\pi}}\frac{3z^2-r^2}{r^2}$
$Y_{2,\pm1}(\theta,\varphi) = \mp\sqrt{\frac{15}{8\pi}}e^{\pm i\varphi}\sin\theta\cos\theta$	$Y_{2,\pm1}(x,y,z) = \mp\sqrt{\frac{15}{8\pi}}\frac{(x\pm iy)z}{r^2}$
$Y_{2,\pm2}(\theta,\varphi) = \sqrt{\frac{15}{32\pi}}e^{\pm 2i\varphi}\sin^2\theta$	$Y_{2,\pm2}(x,y,z) = \mp\sqrt{\frac{15}{32\pi}}\frac{x^2-y^2\pm 2ixy}{r^2}$

Pauli matrices: $\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, $\sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$, $\sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

ELECTROSTATICS

$$\oiint_S \mathbf{E}\cdot d\mathbf{A} = \frac{q_{\text{enc}}}{\varepsilon_0} \qquad \mathbf{E} = -\nabla V \qquad \int_{\mathbf{r}_1}^{\mathbf{r}_2}\mathbf{E}\cdot d\boldsymbol{\ell} = V(\mathbf{r}_1) - V(\mathbf{r}_2) \qquad V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Work done $W = -\int_{\mathbf{a}}^{\mathbf{b}} q\mathbf{E}\cdot d\boldsymbol{\ell} = q\left[V(\mathbf{b}) - V(\mathbf{a})\right]$ Energy stored in elec. field: $W = \frac{1}{2}\varepsilon_0\int_V E^2 d\tau = Q^2/2C$

Relative permittivity: $\varepsilon_{\text{r}} = 1 + \chi_{\text{e}}$

Capacitance in vacuum

Parallel-plate: $C = \varepsilon_0 \dfrac{A}{d}$

Spherical: $C = 4\pi\varepsilon_0 \dfrac{ab}{b-a}$ $(b > a)$

Cylindrical: $C = 2\pi\varepsilon_0 \dfrac{L}{\ln(b/a)}$ $(b > a;$ for a length $L)$

MAGNETOSTATICS

Relative permeability: $\mu_{\mathrm{r}} = 1 + \chi_{\mathrm{m}}$

Lorentz Force: $\mathbf{F} = q\mathbf{E} + q(\mathbf{v} \times \mathbf{B})$

Current densities: $I = \int \mathbf{J} \cdot d\mathbf{A}$, $I = \int \mathbf{K} \cdot d\ell$

Biot-Savart Law: $\mathbf{B}(\mathbf{r}) = \dfrac{\mu_0}{4\pi} \int \dfrac{I d\ell \times \hat{\mathbf{R}}}{R^2}$ ($\mathbf{R}$ is vector from source point to field point $\mathbf{r}$)

For straight wire segment: $B = \dfrac{\mu_0 I}{4\pi s}\left[\sin\theta_2 - \sin\theta_1\right]$ where s is the perpendicular distance from wire.

Infinitely long solenoid: B-field inside is $B = \mu_0 n I$ (n is number of turns per unit length)

Ampere's law: $\oint \mathbf{B} \cdot d\ell = \mu_0 I_{\text{enclosed}}$

Magnetic dipole moment of a current distribution is given by $\mathbf{m} = I \int d\mathbf{a}$.

Force on magnetic dipole: $\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$

Torque on magnetic dipole: $\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}$

B-field of magnetic dipole: $\mathbf{B}(\mathbf{r}) = \dfrac{\mu_0}{4\pi} \dfrac{1}{r^3}\left[3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\right]$

Maxwell's Equations in vacuum

1. $\nabla \cdot \mathbf{E} = \dfrac{\rho}{\varepsilon_0}$ Gauss' Law
2. $\nabla \cdot \mathbf{B} = 0$ no magnetic charge
3. $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ Faraday's Law
4. $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \varepsilon_0 \mu_0 \dfrac{\partial \mathbf{E}}{\partial t}$ Ampere's Law with Maxwell's correction

Maxwell's Equations in linear, isotropic, and homogeneous (LIH) media

1. $\nabla \cdot \mathbf{D} = \rho_f$ — Gauss' Law
2. $\nabla \cdot \mathbf{B} = 0$ — no magnetic charge
3. $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ — Faraday's Law
4. $\nabla \times \mathbf{H} = \mathbf{J}_f + \dfrac{\partial \mathbf{D}}{\partial t}$ — Ampere's Law with Maxwell's correction

Induction

Alternative way of writing Faraday's Law: $\oint \mathbf{E} \cdot d\boldsymbol{\ell} = -\dfrac{d\Phi_B}{dt}$

Mutual and self inductance: $\Phi_2 = M_{21} I_1$, and $M_{21} = M_{12}$; $\Phi = LI$

Energy stored in magnetic field: $W = \frac{1}{2}\mu_0^{-1} \int_V B^2 d\tau = \frac{1}{2} L I^2 = \frac{1}{2} \oint \mathbf{A} \cdot \mathbf{I}\, d\ell$

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence : $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

Curl : $\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$

Laplacian : $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$

Divergence : $\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$

Curl :

$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence : $\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$

Curl : $\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$

Laplacian : $\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

INTEGRALS

$f(x)$	$\int_0^\infty f(x)\,dx$
e^{-ax^2}	$\dfrac{\sqrt{\pi}}{2\sqrt{a}}$
xe^{-ax^2}	$\dfrac{1}{2a}$
$x^2e^{-ax^2}$	$\dfrac{\sqrt{\pi}}{4a^{3/2}}$
$x^3e^{-ax^2}$	$\dfrac{1}{2a^2}$
$x^4e^{-ax^2}$	$\dfrac{3\sqrt{\pi}}{8a^{5/2}}$
$x^5e^{-ax^2}$	$\dfrac{1}{a^3}$
$x^6e^{-ax^2}$	$\dfrac{15\sqrt{\pi}}{16a^{7/2}}$

$$\int_0^\infty \frac{1}{1+ay^2}dy = \pi / 2a^{1/2}$$

$$\int_0^\infty y^n e^{-ay} dy = \frac{n!}{a^{n+1}}$$

$$\int \frac{r^3 dr}{(x^2+r^2)^{3/2}} = (r^2+x^2)^{1/2} + \frac{x^2}{(r^2+x^2)^{1/2}}$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) \qquad \left|\tan^{-1}\left(\frac{x}{a}\right)\right| < \frac{\pi}{2}$$

$$\int \frac{x\,dx}{a^2+x^2} = \frac{1}{2}\ln\left(a^2+x^2\right)$$

$$\int \frac{dx}{x(a^2+x^2)} = \frac{1}{2a^2}\ln\left(\frac{x^2}{a^2+x^2}\right)$$

$$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right)$$

$$= -\frac{1}{ab}\coth^{-1}\left(\frac{ax}{b}\right)$$

$$= -\frac{1}{ab}\tanh^{-1}\left(\frac{ax}{b}\right), \quad a^2x^2 < b^2$$